

Integrable geodesic flows on Riemannian manifolds: Construction and Obstructions *

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Abstract

This paper is a review of recent and classical results on integrable geodesic flows on Riemannian manifolds and topological obstructions to integrability. We also discuss some open problems.

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1 Introduction

1.1 Geodesics on Riemannian manifolds

We start with basic definitions and the statement of the problem to which our paper is devoted.

The first notion is a *geodesic line* on a manifold. To introduce it, consider the following problem. Imagine a point moving on a two-dimensional surface in three-dimensional space. What is its trajectory, if there is no external force acting on the point? The classical mechanics gives the following answer: the point will move with velocity of constant absolute value and in such a way that the acceleration (as a vector in \mathbb{R}^3) is always orthogonal to the surface. The trajectory of the point is called a *geodesic line*. It is not hard to verify that such a motion is described by a system of two second-order differential equations. A less obvious fact is that the geodesics are not changed under transformations which do not touch the interior geometry of the surface, or, in the language of differential geometry, the induced Riemannian metric.

The other approach to define geodesics uses the interior geometry from the very beginning. Consider the following problem. Given two points on a surface, find the shortest curve on the surface connecting them. Such a curve (if it exists) is just a geodesic. In general, there may exist several such curves or no one. However, if the points are sufficiently close, then the desired curve always exists and is unique. Thus the geodesics can be characterized as locally shortest lines on the surface. This interpretation allows one to define geodesics on an arbitrary manifold endowed with a Riemannian metric.

However, the first approach can be applied for the general case as well. Geodesics are the trajectories of points moving by inertia. In other words, the acceleration of a point must be identically zero. One should only explain that in this case by the acceleration we mean the *covariant* derivative of the velocity vector. Let us recall that such a derivative is naturally defined on every Riemannian manifold and it is called *Levi-Civita connection*.

It can be easily shown that the geodesics are described by a system of second-order differential equations which can be written in the Hamiltonian form. Let M be a smooth manifold with a Riemannian metric $g = (g_{ij})$. Consider an arbitrary local coordinate system x^1, \dots, x^n and pass from velocities \dot{x}^i to

momenta p_j by using the standard transformation $p_j = g_{ij}\dot{x}^i$. Then in the new coordinates x^i, p_i ($i = 1, \dots, n$) the equations of geodesics read:

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}, \quad (1.1)$$

where H (the Hamiltonian) is interpreted as the kinetic energy:

$$H = \frac{1}{2} \sum_{i,j=1}^n g^{ij} p_i p_j = \frac{1}{2} \sum_{i,j=1}^n g_{ij} \dot{x}^i \dot{x}^j.$$

Here g^{ij} are the coefficients of the tensor inverse to the metric.

This system of equations is Hamiltonian on the cotangent bundle T^*M (with the standard symplectic form $\omega = \sum dp_i \wedge dx^i$) and is called *the geodesic flow* of the Riemannian manifold (M, g) . Speaking more precisely, the geodesic flow is the one-parameter group of diffeomorphisms defined by this system of differential equations.

1.2 Integrable geodesic flows

We will be interested in the global behavior of geodesics on closed Riemannian manifolds. Namely we want to distinguish the case of the so-called integrable geodesic flows. Before giving the formal definition, we consider some examples.

First take the two-dimensional sphere. The geodesics on it are well known: these are equators. All of them are closed and of the same length. Thus, the dynamics of the geodesic flow is very simple. The second example are geodesics on the surface of revolution. In a typical case the behavior of a geodesic is as follows: it moves on the surface around the axis of revolution and at the same time oscillates along this axis. If the geodesic is not closed, then its closure is an annulus-like region on the surface. As we see, the dynamics is quite regular and is a superposition of rotation and oscillation. An analogous picture can be seen on the three-axial ellipsoid. This case is more complicated. Here there are two kinds of annulus-like regions inside which geodesics move. But in the whole, the behavior of geodesics is still regular. Now consider an arbitrary closed surface without any special symmetries (for example, deform a little bit the standard sphere). We shall see that the behavior of geodesics loses regularity and becomes chaotic.

Regular behaviour is the characteristic property of integrable geodesic flows which are formally defined as follows.

Definition 1.1 The geodesic flow (1) is called *completely Liouville integrable*, if it admits n smooth functions $f_1(x, p), \dots, f_n(x, p)$ satisfying three conditions:

- 1) $f_i(x, p)$ is an integral of the geodesic flow, i. e., is constant along each geodesic line $(x(t), p(t))$;
- 2) f_1, \dots, f_n pairwise commute with respect to the standard Poisson bracket on T^*M , i.e., $\{f_i, f_j\} = \sum_{\alpha} \left(\frac{\partial f_i}{\partial x^{\alpha}} \frac{\partial f_j}{\partial p_{\alpha}} - \frac{\partial f_j}{\partial x^{\alpha}} \frac{\partial f_i}{\partial p_{\alpha}} \right) = 0$;
- 3) f_1, \dots, f_n are functionally independent on T^*M .

Remark 1.1 The third condition needs to be commented. The functional independence of the integrals can be meant in three different senses. The differentials of f_1, \dots, f_n must be linearly independent:

- a) on an open everywhere dense subset,
- b) on an open everywhere dense subset of full measure,
- c) everywhere except for a piece-wise smooth polyhedron.

Instead of these conditions one can assume all the functions to be real analytic. Then it suffices to require their functional independence at least at one point (we will call such a situation *analytic integrability*).

The regularity of dynamics in the case of integrability follows from the following classical Liouville theorem (see [4]):

Theorem 1.1 *Let $X^n = \{f_1 = c_1, \dots, f_n = c_n\}$ be a common level surface of the first integrals of a Hamiltonian system. If this surface is regular (i.e., the differentials of f_1, \dots, f_n are independent on it), compact and connected, then*

- 1) X^n is diffeomorphic to the n -dimensional torus;
- 2) the dynamics on this torus is quasi-periodic, i.e., can be linearized in appropriate angle coordinates $\varphi_1, \dots, \varphi_n$:

$$\varphi_1(t) = \omega_1 t, \dots, \varphi_n(t) = \omega_n t.$$

Thus, except a certain singular set, the phase space of the system turns out to be foliated into invariant tori with quasi-periodic dynamics. However the dynamics on the singular set may be rather complicated (see below).

1.3 Statement of the problem

The general question discussed below can be formulated as follows: Which closed smooth manifolds admit Riemannian metrics with integrable geodesic flows? In other words, we want to divide all manifolds into two classes depending on the fact if there exist or not integrable geodesic flows on them. This question is rather complicated and so far we cannot expect any complete answer. At present there are, in essence, only two general approaches to the problem.

The positive answer to the existence question is strictly individual: we can take a certain manifold (or a certain class of manifolds) and construct on it an explicit example of a metric with integrable geodesic flow. So far there is no other method except the explicit construction. Thus the problem is reduced to studying new constructions of integrable Hamiltonian systems in the particular case of geodesic flows. The second approach deals with topological obstructions to integrability. More precisely, the problem is to find a topological property of a manifold which *is not compatible* with integrability. Then all the manifold having such a property belong to the second class, i.e., admit no integrable geodesic flows.

Notice that the character of first integrals is very important. Usually one consider the integrals of three different types: C^∞ -smooth, analytic or polynomials in momenta. In the last case one can fix or bound their degrees. Besides, sometimes one has to require some topological restrictions for the structure of the singular set. Each time we deal with a specific problem and obtain a new result, the most important of which we shall try to mention below.

Let us emphasize that in our paper we deal with topological obstructions to integrability only. This means that we are mostly interested in the principal possibility of constructing integrable systems on a given manifold. There is another problem in Hamiltonian dynamics very natural and important, namely,

the problem of analytical obstructions to integrability, which can be formulated in a very general way as follows. Given a Hamiltonian function H , is the corresponding Hamiltonian system integrable or not? There are many quite powerful methods to answer this question (see Kozlov [46, 47]). One of the most famous and classical is, for example, the Painlevé-Kovalevskaya test. Among modern approaches to this problem one should mention the papers by Ziglin [86], Yoshida [84, 85], and Ruiz and Ramis (e.g., see [72]).

Speaking of examples of integrable geodesic flows, in our paper we will confine ourselves to the existence problem. However the complete description of integrable geodesic flows of a certain type is extremely interesting and deep problem. The algebro-geometric approach to the classification of integrable geodesic flows has been developed by Adler and Van Moerbeke [1, 2] (case of $SO(4)$) and Haine [36] (case of $SO(n)$).

2 Classical examples of integrable geodesic flows

Geodesic flows on two-dimensional surfaces. Classical examples of surfaces with integrable geodesic flows are the two-dimensional sphere $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$ of constant curvature and the flat torus. Geodesics on the sphere are equators (i.e., sections by the planes passing through its center). Since all geodesics are closed, in this case there are three independent first integrals instead of two. As a "basis" one can take linear integrals corresponding to infinitesimal rotations of the sphere. For example, the integral corresponding to the rotation about the axis Ox_3 has the form

$$f_3(x, p) = p(\xi_3(x)),$$

where $p \in T_x^*S^2$, and $\xi_3 = \partial/\partial\varphi$ is the vector field related to the standard spherical coordinate φ on S^2 : $\xi_3(x_1, x_2, x_3) = (-x_2, x_1, 0)$. In terms of the tangent bundle these integrals admit the following natural representation as one vector integral:

$$F(x, \dot{x}) = (f_1, f_2, f_3) = [x, \dot{x}]$$

where $x \in S^2 \subset \mathbb{R}^3$ and $\dot{x} \in T_x S^2$ are considered as vectors of three-dimensional Euclidean space \mathbb{R}^3 . The geometrical meaning of F is clear: this is the vector orthogonal to the plane to which the geodesic belongs. These three linear integrals do not commute, but as two commuting integrals one can take any of them and the Hamiltonian of the geodesic flow $H = \frac{1}{2}(f_1^2 + f_2^2 + f_3^2)$.

In the case of a flat metric $ds^2 = dx_1^2 + dx_2^2$ on the torus, the geodesics are quasiperiodic windings $x_i(t) = c_i t$ ($i = 1, 2$), where $x_1 \bmod 2\pi$, $x_2 \bmod 2\pi$ denote standard angle coordinates. The first commuting integrals are the corresponding momenta p_1 and p_2 . The Hamiltonian of the flow is expressed by p_1 and p_2 in the obvious way: $H(x_1, x_2, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2)$, or more generally, $H(x_1, x_2, p_1, p_2) = \frac{1}{2}(ap_1^2 + 2bp_1p_2 + cp_2^2)$.

The next classical examples are metrics of revolution and Liouville metrics.

Theorem 2.1 (Clairaut) *The geodesic flow on a surface of revolution admits a non-trivial linear integral and, consequently, is integrable.*

The Clairaut integral has the same form as f_3 in the case of the sphere (if we consider a surface of revolution about the axis Ox_3). It admits also the following geometrical description. Let ψ be the angle between the geodesic and

the parallel on the surface of revolution, r be the distance to the revolution axis. Then the Clairaut integral is $r \cos \psi$.

The existence of such an integral is the reflection of the following classical result (Noether theorem): let a Riemannian metric g_{ij} admit a one-parameter isometry group $\phi^s : M \rightarrow M$. Then the corresponding geodesic flow has the linear integral of the form $f_\xi(x, p) = p(\xi(x))$, where $\xi(x) = \frac{d}{ds}|_{s=0} \phi^s(x)$ is the vector field associated with the one-parameter group ϕ^s (that is, the corresponding infinitesimal isometry). More generally: if a metric admits an isometry group G , then the corresponding geodesic flow has an algebra of first integrals which is isomorphic to the Lie algebra of the group G .

Theorem 2.2 (Liouville) *The geodesic flow of the metric*

$$ds^2 = (f(x_1) + g(x_2))(dx_1^2 + dx_2^2) \quad (2.1)$$

admits the non-trivial quadratic integral of the form

$$F(x, p) = \frac{g(x_2)p_1^2 - f(x_1)p_2^2}{f(x_1) + g(x_2)} \quad (2.2)$$

and, consequently, is integrable.

This formula, in general, is local. It is not hard to construct an example of a metric on a two-dimensional surface, which admits representation (2.1) in some local coordinates at each point, but on the whole surface, the integrals given by (2.2) cannot be arranged in one globally defined integral.

One of such examples is the constant negative curvature metric on a surface of genus $g > 1$. Locally its geodesic flow admits a quadratic integral (speaking more precisely, there are three independent linear integrals from which one can combine a quadratic one). However, there are no such global integrals (see the section below).

Projectively equivalent metrics. There is another interesting class of manifolds closely related to our problem. These are manifolds admitting two *projectively* (or *geodesically*) equivalent Riemannian metrics g and \tilde{g} , that is, metrics having the same geodesics (considered as unparametrized curves) (see Levi-Civita [49]). If g and \tilde{g} are in general position, i.e., there exists at least one point at which the operator $g\tilde{g}^{-1}$ has simple spectrum, then the geodesic flows of the both metrics are integrable. Moreover, the first integrals are all quadratic or linear functions. A rather elegant proof of this fact has been obtained by V. Matveev and P. Topalov [52, 79] (it is interesting that the corresponding Hamilton-Jacobi equations admit separation of variables and that the problem can be considered from a bi-Hamiltonian point of view, see [18]). The open problem is to describe the class of all such manifolds.

Geodesics on the ellipsoid. Finally, as one of the most beautiful examples we have the geodesic flow of the standard metric $ds^2 = \langle dx, dx \rangle$ on the $(n-1)$ -dimensional ellipsoid [62, 63, 43, 30]

$$Q = \{ \langle x, A^{-1}x \rangle = 1 \} \subset \mathbb{R}^n, \quad A = \text{diag}(a_1, \dots, a_n), \quad a_1 > a_2 > \dots > a_n.$$

The problem was solved by Jacobi by separation of variables in elliptic coordinates [37]. Moser, in his famous papers [62, 63], found an L-A pair representation and commuting integrals in Euclidian coordinates by using geometry of quadrics.

Theorem 2.3 (Moser, 1980) *The functions F_k :*

$$F_k(x, \dot{x}) = \dot{x}_k^2 + \sum_{l \neq k} \frac{(x_k \dot{x}_l - x_l \dot{x}_k)^2}{a_k - a_l}, \quad k = 1, \dots, n,$$

restricted to the tangent bundle $TQ\{x, \dot{x}\}$ Poisson commute and give complete integrability of the geodesic flow on the ellipsoid Q .

The geometric interpretation of the integrability is described by the Chasles theorem: the tangent line of a geodesic $x(t)$ on Q is also tangent to a fixed set of confocal quadrics $Q(\alpha_i) = \{((A - \alpha_i Id)^{-1}x, x) = 1\}$, $\alpha_i \in \mathbb{R}$, $i = 1, \dots, n-2$ (for example, see [43, 30]).

3 Topological obstructions to integrability

3.1 Case of two-dimensional surfaces

The first results on topological obstructions to integrability relate to the case of two-dimensional surfaces.

Theorem 3.1 (Kozlov, 1979 [45]) *Two-dimensional surfaces of genus $g > 1$ admit no analytically integrable geodesic flows.*

The original proof essentially used some delicate properties of analytic functions. But later it was understood that the analyticity condition could be essentially weakened (see Taimanov [76, 77]). However, it is still not clear if it is possible to omit any additional conditions to the first integrals.

Question 3.1 *Given a two-dimensional surface of genus $g > 1$, do there exist integrable geodesic flows on it with C^∞ -smooth integrals?*

Here is another result related to the same class of surfaces.

Theorem 3.2 (Kolokol'tsov, 1982 [45]) *Two-dimensional surfaces of genus $g > 1$ admit no geodesic flows integrable by means of an integral polynomial in momenta (the coefficients of this polynomial are assumed to be smooth functions without any analyticity conditions).*

The idea of the proof of Theorem 3.2 is rather different from that of Kozlov's theorem. By using the polynomial integral one constructs a certain holomorphic form on the given surface. Then analyzing its zeros and poles one can estimate the genus of the surface.

Question 3.2 *Is there a multidimensional analog of Kolokol'tsov theorem? In other words, are there topological obstructions to polynomial integrability?*

If we confine ourselves to linear and quadratic integrals, then the problem is getting simpler and it is, probably, possible to obtain the complete list of manifolds with linearly and quadratically integrable geodesic flows. The point is that under some additional conditions such flows admit separation of variables on the configuration space. These variables will have, however, certain singularities and the problem is reduced to studying the topology of manifolds admitting global coordinate systems with special types of singularities. Such an approach has been used by Kiyohara [41], but the final answer is not yet obtained.

3.2 Topological obstructions in the case of non simply connected manifolds

The next fundamental result is due to I. Taimanov [76, 77].

Theorem 3.3 (Taimanov, 1987) *If a geodesic flow on a closed manifold M is analytically integrable, then*

- 1) *the fundamental group of M is almost commutative (i.e., contains a commutative subgroup of finite index);*
- 2) *if $\dim H_1(M, \mathbb{Q}) = d$, then $H^*(M, \mathbb{Q})$ contains a subring isomorphic to the rational cohomology ring of the d -dimensional torus;*
- 3) *if $\dim H_1(M, \mathbb{Q}) = n = \dim M$, then the rational cohomology rings of M and of the n -dimensional torus are isomorphic.*

The idea of the proof is purely topological and the analyticity condition is not essential. In fact, I. Taimanov proved this result under the much weaker assumption that a geodesic flow is geometrically simple. This means that the structure of the singular set where the first integrals are dependent is not too complicated from the topological point of view. Speaking more precisely, to this singular set one should add some new "cuts" in such a way that the rest becomes a trivial fibration into Liouville tori over a disjoint union of discs. Besides the geometric simplicity condition takes into account some properties of the projection of this "completed" singular set from the cotangent bundle onto the configuration space (see [76, 77]).

3.3 Topological entropy and integrability of geodesic flows

In 1991 Paternain suggested an approach to finding topological obstructions to integrability of geodesic flows based on the notion of *topological entropy* [67, 68]. The topological entropy is a characteristic of a dynamical system on a compact manifold, which measures, in a certain sense, its chaoticity. Since, as a rule, integrable Hamiltonian systems have zero topological entropy, one can proceed as follows. First one may try to estimate the topological entropy of a geodesic flow on a given manifold by purely topological means. Very often one can do that even without any information about the riemannian metric: if the topology of a manifold is sufficiently complicated (see examples below), then the topological entropy of any geodesic flow has automatically to be positive. The second part of the problem is to prove that the integrability of a geodesic flow implies indeed vanishing of the topological entropy (perhaps under some additional conditions to the first integrals).

Recall the definition of topological entropy. Let F^t be a dynamical system on a compact manifold X considered as a one-parameter group of diffeomorphisms. Suppose that we want to approximate this systems up to $\varepsilon > 0$ on a segment $[0, T]$ by using only finite number of solutions. In other words, we want to choose a finite number of points $x_1, \dots, x_{N(\varepsilon, T)}$ in such a way that for any other point $y \in X$ there exists x_i satisfying

$$\text{dist}(F^t(y), F^t(x_i)) < \varepsilon$$

for any $t \in [0, T]$. Here dist denotes any metric compatible with the topology of X . Suppose now that $N(\varepsilon, T)$ is the minimal number of such points x_i and consider its asymptotics as $\varepsilon \rightarrow 0$ and $T \rightarrow \infty$.

Definition 3.1 The topological entropy of the flow F^t is define to be

$$h_{top}(F^t) = \lim_{\varepsilon \rightarrow 0} \limsup_{T \rightarrow \infty} \frac{\ln N(\varepsilon, T)}{T}.$$

The next theorem is the first result showing how the topology of a manifold affect the topological entropy of geodesic flows.

Theorem 3.4 (Dinaburg, 1971 [26]) *If the fundamental group $\pi_1(M)$ of M has exponential growth, then the topological entropy of the geodesic flow is positive for any smooth Riemannian metric on M . In particular, the topological entropy of any geodesic flow on a two-dimensional surface of genus $g > 1$ is positive.*

Recall that the topological entropy of a geodesic flow is that of its restriction onto a compact isoenergy surface $Q^{2n-1} = \{H(x, p) = 1\} \subset T^*M^n$, $H(x, p) = \frac{1}{2}|p|^2$.

It is important that the entropy approach works successfully in the case of simply-connected manifolds when Kozlov's and Taimanov's theorems cannot be applied.

Theorem 3.5 (Paternain, 1991 [67, 68]) *If a simply-connected manifold is not rationally elliptic, then the topological entropy of any geodesic flow on it is positive.*

The rational ellipticity means that the rational homotopy groups of M are trivial starting from a certain dimension N , i.e., $\pi_n(M) \otimes \mathbb{Q} = 0$ for any $n > N$. The next result not only guarantees the positiveness of the topological entropy for a large class of manifolds, but also allows one to estimate it from below.

Theorem 3.6 (Babenko, 1997 [6]) *The topological entropy of a geodesic flow on a simply connected Riemannian manifold (M, g) admits the following estimate*

$$h_{top}(g) \geq D^h(M, g)^{-1} \limsup_{n \rightarrow \infty} \frac{\ln(\text{rank } \pi_n(M))}{n},$$

where $D^h(M, g)$ is the homology diameter of (M, g) .

Note that the limit $\limsup_{n \rightarrow \infty} \frac{\ln(\text{rank } \pi_n(M))}{n}$ is equal to zero only for rationally elliptic manifolds, otherwise this is always a positive number. The homology diameter of a manifold depend on the choice of g and is, therefore, a geometric characteristic of a manifold (exact definition can be found in [6]).

To use the topological entropy as an obstruction to integrability, it was necessary to show vanishing topological entropy for integrable geodesic flows. Under some rather strong additional assumptions this statement holds indeed.

Theorem 3.7 (Paternain, 1991 [67, 68]) *Let (M, g) be a smooth compact Riemannian manifold. Suppose that the geodesic flow on it is integrable in the class of non-degenerate first integrals. Then the topological entropy of this flow vanishes.*

The non-degeneracy of integrals means the following. First consider the case when x is a singular point for all the first integrals of the system. Then the non-degeneracy is equivalent to the existence of such an integral f that x is non-degenerate in the usual sense of the Morse theory, that is, $\det \left(\frac{\partial^2 f}{\partial x^i \partial x^j} \right) \neq 0$. If

some integrals have non-zero differentials at x , then we make local symplectic reduction by the action of these integrals after which we get to the previous situation. The point x is called non-degenerate for the initial system if it is such for the reduced one. The integrals of a Hamiltonian system are called non-degenerate if every point x of a symplectic manifold is non-degenerate.

However, the non-degeneracy assumption is rather strong and, in the multidimensional case, holds very rarely. For example, the degeneracy appears in such a natural case as the geodesic flow on the n -dimensional ellipsoid ($n \geq 3$). One of the reasons is the existence of stable degenerate singularities which cannot be avoided by small perturbation and, consequently, are generic.

4 Counterexamples

It has been, however, understood recently that additional assumptions on the first integrals cannot be completely omitted. In other words, in general case neither topological entropy, nor "complexity" of the fundamental group is an obstruction to integrability of geodesic flows.

The first interesting example of an "exotic" integrable flow was constructed by L. Butler [24].

Theorem 4.1 (Butler, 1998) *There is a three-dimensional Riemannian (real-analytic) NIL-manifold (M^3, g) such that*

- 1) *the geodesic flow of the metric g on M^3 is completely integrable by means of C^∞ -smooth first integrals (moreover, two of these integrals are real-analytic functions);*
- 2) *the fundamental group $\pi_1(M)$ is not almost commutative and has polynomial growth;*
- 3) *the topological entropy of the geodesic flow vanishes.*

This example shows that in the smooth case the statement analogous to Taimanov's theorem (Theorem 3.3) fails. Thus, the geometric simplicity assumption introduced by Taimanov is really very important (Butler's example is not geometrically simple). Besides this is the first example of the situation when a real-analytic flow is integrable, but its integrals cannot be real-analytic functions.

The topological structure of M^3 in Butler's example is quite simple. This is a fibration $M^3 \xrightarrow{T^2} S^1$ over the circle with the torus as a fiber and with the monodromy matrix of the form

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

In other words, to reconstruct the manifold one needs to take the direct product $T^2 \times [0, 2\pi]$ and then to glue its feet $T^2 \times \{0\}$ and $T^2 \times \{2\pi\}$ by the linear map given by the matrix A in angle coordinates.

Using the idea of Butler, a year later the first author and Taimanov in [13] constructed an example of a three-dimensional manifold with integrable geodesic flow having positive topological entropy.

Theorem 4.2 (Bolsinov, Taimanov, 1999) *There is a three-dimensional Riemannian (real-analytic) SOL-manifold (M^3, g) such that*

- 1) the geodesic flow of the metric g on M^3 is completely integrable by means of C^∞ -smooth first integrals (moreover, two of these integrals are real-analytic functions);
- 2) the fundamental group $\pi_1(M)$ is not almost commutative and has exponential growth;
- 3) the topological entropy of the geodesic flow is positive.

In this example, M^3 has "almost the same" structure as in Butler's example: it suffices to replace the matrix A by an integer hyperbolic matrix, for instance,

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The Riemannian metric g on M^3 is described as follows. Let x, y, z be local coordinates on M , where $z \bmod 2\pi$ is an angle coordinate on the base S^1 , and $x \bmod 2\pi, y \bmod 2\pi$ are angle coordinates on the fiber T^2 . Then

$$ds^2 = d\tilde{s}_z^2 + dz^2,$$

where

$$d\tilde{s}_z^2 = a_{11}(z)dx^2 + 2a_{12}(z)dxdy + a_{22}(z)dy^2$$

is a flat metric on the fiber T_z^2 over $z \in S^1$. The coefficients a_{11}, a_{12}, a_{22} are chosen so that the metric turns out to be smooth on the whole manifold M .

Since the coefficients of the metric do not depend on x and y , the corresponding momenta p_x, p_y are commuting first integrals of the geodesic flow. The only problem is that p_x and p_y are not globally defined on M^3 because of non-triviality of the monodromy of the T^2 -fibration. However, it is possible to construct other functions $F_1 = F_1(p_x, p_y)$ and $F_2 = F_2(p_x, p_y)$ which will be preserved under the monodromy action. As such functions one should take invariants of the action of the cyclic group \mathbb{Z} generated by the linear transformation $B^{-1\top}$ on the two-dimensional space $\mathbb{R}^2(p_x, p_y)$. It is interesting to remark that the orbit structure of this action is such that only one of the invariants is an analytic function. The second can be chosen C^∞ -smooth, but not real-analytic. The functions F_1 and F_2 (together with the Hamiltonian) guarantee the complete integrability of the geodesic flow on (M, g) .

The positiveness of the topological entropy easily follows from Dinaburg's theorem, but also can be explained directly. The point is that the geodesic flow admits a natural invariant manifold N formed by the geodesics of the form $x(t) = \text{const}, y(t) = \text{const}, z(t) = t$. It is easily seen that the union of such geodesics is a submanifold in T^*M diffeomorphic to the base M . The geodesic flow ϕ^t restricted onto N preserves the T^2 -fibers. Moreover the 2π -shift along the flow transforms each fiber into itself $\phi^{t=2\pi} : T^2 \rightarrow T^2$ and coincides with the hyperbolic automorphism of the torus given by the matrix B . It is well-known (as one of basic examples) that the topological entropy of such an automorphism is positive and equals to $\ln \lambda$, where λ is the maximal eigenvalue of B (see, for example, [40]). Thus the geodesic flow admits a subsystem with positive topological entropy and, consequently, possesses this property itself.

The constructions of Theorems 4.1, 4.2 are naturally generalized to the case of arbitrary dimension [14, 25].

Note that the topological structure of the singular set in these examples is quite simple. This is a finite polyhedron whose strata are not just only smooth

but also real-analytic submanifolds. "Non-analytic" is the way of how Liouville tori approach this singular set.

It is worth to explain why the above geodesic flows are not geometrically simple in the sense of Taimanov [76, 77]. The point is that the base of the foliation into Liouville tori is not simply connected. To make it such (as required in the definition of geometric simplicity) we need additional cuts of the base. But this is impossible to do by a "geometrically simple" way: in each tangent space such a cut will consist of infinitely many two-dimensional planes.

The above examples lead to the following questions:

Question 4.1 *Which additional properties of the first integrals guarantee vanishing the topological entropy?*

Question 4.2 *Is analyticity such a condition? In other words, is the topological entropy an obstruction to analytic integrability?*

Question 4.3 *Is it possible to include an arbitrary (as chaotic as one wishes) dynamical system as a subsystem into an integrable Hamiltonian system of higher dimension?*

5 Geodesic flows on homogeneous spaces and bi-quotients of Lie groups

Since the integrability is closely related to the existence of some symmetries (possibly hidden), for the construction of multi-dimensional integrable examples, as a rule we should use metrics with large symmetry groups. In the final construction the symmetry can be removed by algebraic modification of metrics.

The classical example is the geodesic flow of a left-invariant metric on the Lie group $SO(3)$. The geodesic flow of such a metric describes the motion of a rigid body about a fixed point under its own inertia. This problem was solved by Euler. In general, the geodesic flow of a left-invariant metrics on a Lie group G after G -reduction reduces to the *Euler equations* on \mathfrak{g}^* (the dual space of the Lie algebra \mathfrak{g}), which are Hamiltonian equations with respect to the Lie-Poisson bracket on \mathfrak{g}^* [4].

A multidimensional generalization of the Euler case has been suggested by Manakov [51]. Using his idea, Mishchenko and Fomenko proposed the argument shift method (see below) and constructed integrable examples of Euler equations for all compact groups [54] and proved the integrability of the original geodesic flows [55, 56].

Theorem 5.1 (Mishchenko, Fomenko 1976) *Every compact Lie group admits a family of left-invariant metrics with completely integrable geodesic flows.*

There are many other important constructions on various Lie algebras (we mention just some of the review papers and books [5, 10, 31, 32, 71, 70]). In particular, the problem of algebraic integrability of geodesic flows on $SO(4)$ and $SO(n)$ is studied in [1, 2, 36].

Throughout the paper, by a *normal* G -invariant Riemannian metric on the homogeneous space G/H of a compact group G , we mean the metric induced from a bi-invariant metric on G .

Theorem 5.2 (Thimm 1981, Mishchenko 1982) *Geodesic flows of normal metrics on compact symmetric spaces G/H are completely integrable.*

These results are generalized by Brailov, Guillemin and Sternberg and Mikityuk. Brailov applied the Mishchenko–Fomenko construction to the symmetric spaces and obtained families of non G -invariant metrics with integrable flows on symmetric spaces [21, 22]. Guillemin and Sternberg [33, 34] and Mikityuk [59] described the class of homogeneous spaces G/H on which all G -invariant Hamiltonian systems are integrable by means of Noether integrals [33, 34, 59]. It appears that in those cases (G, H) is a spherical (or Gelfand) pair.

If G/H is a symmetric space then (G, H) is a spherical pair, but there exist spherical pairs which are not symmetric. Note that, for G compact, (G, H) is a spherical pair if and only if G/H is a weekly symmetric space (see [81] and references therein).

Examples of homogeneous, non (weekly) symmetric spaces G/H with integrable geodesic flows are given by Thimm [78], Paternain and Spatzier [69] and Mikityuk and Stepin [61]. Also, to this list now we can add the above mentioned examples constructed by Butler [24] and Taimanov and the first author [13, 14].

Particular examples of non-homogeneous manifolds (bi-quotients of compact Lie groups) with integrable geodesic flows are obtained by Paternain and Spatzier [69] and Bazaikin [7].

It appears that many of those results can be considered together within the framework of non-commutative integrability. This approach allowed us to obtain general theorems on integrability of geodesic flows on homogeneous spaces and bi-quotients of compact Lie groups [15, 16].

5.1 Non-commutative integrability

There are a lot of examples of integrable Hamiltonian systems with n degrees of freedom that admit more than n (noncommuting) integrals. Then n dimensional Lagrangian tori are foliated by lower dimensional isotropic tori (sometimes such systems are called *superintegrable*). The concept of non-commutative integrability has been introduced by Mishchenko and Fomenko [55, 56] (see also [64, 20, 31]).

Let M be a $2n$ -dimensional symplectic manifold. Let $(\mathcal{F}, \{\cdot, \cdot\})$ be a Poisson subalgebra of $(C^\infty(M), \{\cdot, \cdot\})$. Suppose that in the neighborhood of a generic point x we can find exactly l independent functions $f_1, \dots, f_l \in \mathcal{F}$ and the corank of the matrix $\{f_i, f_j\}$ is equal to some constant r . Then numbers l and r are called *differential dimension* and *differential index* of \mathcal{F} and they are denoted by $\text{ddim } \mathcal{F}$ and $\text{dind } \mathcal{F}$, respectively. The algebra \mathcal{F} is called *complete* if:

$$\text{ddim } \mathcal{F} + \text{dind } \mathcal{F} = \dim M.$$

In fact, this definition is equivalent to any of the following three conditions.

Take a generic point x and l independent functions $f_1, \dots, f_l \in \mathcal{F}$ in some small neighborhood U of x . Then

- (i) Common level sets of f_1, \dots, f_l in U are isotropic.
- (ii) The subspace generated by $df_1(x), \dots, df_l(x)$ is coisotropic in T_x^*M .
- (iii) If there is a function f which commute with \mathcal{F} on U then $f = f(f_1, \dots, f_l)$ on U .

In particular, we shall say that the algebra \mathcal{F} is *complete at x* if one of the conditions (i–iii) is satisfied.

The Hamiltonian system $\dot{x} = X_H(x)$ is *completely integrable in the non-commutative sense* if it possesses a complete algebra of first integrals \mathcal{F} . Then each connected compact component of a regular level set of the functions $f_1, \dots, f_l \in \mathcal{F}$ is an r -dimensional invariant isotropic torus \mathbb{T}^r (see [55, 64, 56, 20, 31]). Similarly as in the Liouville theorem, in a neighborhood of \mathbb{T}^r there are *generalized action-angle variables* p, q, I, φ , defined in a toroidal domain $\mathcal{O} = \mathbb{T}^r \{\varphi\} \times B_\sigma \{I, p, q\}$,

$$B_\sigma = \{(I_1, \dots, I_r, p_1, \dots, p_k, q_1, \dots, q_k) \in \mathbb{R}^l, \sum_{i=1}^r I_i^2 + \sum_{j=1}^k q_j^2 + p_j^2 \leq \sigma^2\}$$

such that the symplectic form becomes $\omega = \sum_{i=1}^r dI_i \wedge d\varphi_i + \sum_{i=1}^k dp_i \wedge dq_i$, and the Hamiltonian function depends only on I_1, \dots, I_r . The Hamiltonian equations take the following form in action-angle coordinates:

$$\dot{\varphi}_1 = \omega_1(I) = \frac{\partial H}{\partial I_1}, \dots, \dot{\varphi}_r = \omega_r(I) = \frac{\partial H}{\partial I_r}, \quad \dot{I} = \dot{p} = \dot{q} = 0.$$

Example 5.1 Consider a Riemannian manifold (Q, ds^2) with closed geodesics: for every $x \in Q$, all geodesics starting from x return back to the same point. Then one can find a complete algebra of integrals \mathcal{F} with $\text{dind } \mathcal{F} = 1$. In other words, the geodesic flow is completely integrable in non-commutative sense [16].

Note that the concept of noncommutative integrability can be naturally extended to Poisson manifolds $(N, \{\cdot, \cdot\})$. The algebra \mathcal{F} is *complete* if

$$\text{ddim } \mathcal{F} + \text{dind } \mathcal{F} = \dim N + \text{corank } \{\cdot, \cdot\},$$

i.e., the restriction of \mathcal{F} to a generic symplectic leaf M of N is complete on M . Also, if a Hamiltonian system $\dot{x} = X_H(x)$ on N possesses a complete algebra of first integrals \mathcal{F} , then (under compactness condition) N is almost everywhere foliated by $(\text{dind } \mathcal{F} - \text{corank } \{\cdot, \cdot\})$ -dimensional invariant tori with quasi-periodic dynamics.

5.2 Integrable geodesic flows on G/H and $K \backslash G/H$

Let a connected compact Lie group G act on a $2n$ -dimensional connected symplectic manifold (M, ω) . Suppose the action is Hamiltonian, i.e., G acts on M by symplectomorphisms and there is a well-defined momentum mapping $\Phi: M \rightarrow \mathfrak{g}^*$ (\mathfrak{g}^* is the dual space of the Lie algebra \mathfrak{g}) such that one-parameter subgroups of symplectomorphisms are generated by the Hamiltonian vector fields of functions $f_\xi(x) = \Phi(x)(\xi)$, $\xi \in \mathfrak{g}$, $x \in M$ and $f_{[\xi_1, \xi_2]} = \{f_{\xi_1}, f_{\xi_2}\}$. Then Φ is equivariant with respect to the given action of G on M and the co-adjoint action of \mathfrak{G} on \mathfrak{g}^* . In particular, if $\mu \in \Phi(M)$ then the co-adjoint orbit $\mathcal{O}_G(\mu)$ belongs to $\Phi(M)$.

Consider the following two natural classes of functions on M . Let \mathcal{F}_1 be the set of functions in $C^\infty(M)$ obtained by pulling-back the algebra $C^\infty(\mathfrak{g}^*)$ by the moment map $\mathcal{F}_1 = \Phi^* C^\infty(\mathfrak{g}^*)$. Let \mathcal{F}_2 be the set of G -invariant functions in $C^\infty(M)$. The mapping $h \mapsto f_h = h \circ \Phi$ is a morphism of Poisson structures:

$$\{f_{h_1}(x), f_{h_2}(x)\} = \{h_1(\mu), h_2(\mu)\}_{\mathfrak{g}^*}, \quad \mu = \Phi(x),$$

where $\{\cdot, \cdot\}_{\mathfrak{g}^*}$ is the Lie-Poisson bracket on \mathfrak{g}^* :

$$\{f(\mu), g(\mu)\}_{\mathfrak{g}^*} = \mu([df(\mu), dg(\mu)]), \quad f, g : \mathfrak{g}^* \rightarrow \mathbb{R}.$$

Thus, \mathcal{F}_1 is closed under the Poisson bracket. Since G acts in a Hamiltonian way, \mathcal{F}_2 is closed under the Poisson bracket as well. In other words, \mathcal{F}_1 and \mathcal{F}_2 are Lie subalgebras in $C^\infty(M)$.

The second essential fact is that $h \circ \Phi$ commute with any G -invariant function (the Noether theorem). In other words: $\{\mathcal{F}_1, \mathcal{F}_2\} = 0$.

The following theorem, although it is a reformulation of some well known facts about the momentum mapping (see [35, 50]), is fundamental in the considerations below. This formulation was suggested by A. S. Mishchenko.

Theorem 5.3 *The algebra of functions $\mathcal{F}_1 + \mathcal{F}_2$ is complete:*

$$\text{ddim}(\mathcal{F}_1 + \mathcal{F}_2) + \text{dind}(\mathcal{F}_1 + \mathcal{F}_2) = \dim M.$$

Let $A \subset C^\infty(\mathfrak{g}^*)$ be a Lie subalgebra. By \mathcal{A} we shall denote the pull-back of A by the moment map: $\mathcal{A} = \Phi^*(A) = \{f_h = h \circ \Phi, h \in A\}$. Let \mathcal{B} be a subalgebra of \mathcal{F}_2 .

Corollary 5.1 [16] *(i) $\mathcal{A} + \mathcal{F}_2$ is a complete algebra on M if and only if*

A is a complete algebra on the generic orbit $\mathcal{O}_G(\mu) \subset \Phi(M)$ of the co-adjoint action.

(ii) If \mathcal{B} is complete (commutative) subalgebra and A is complete (commutative) algebra on the orbit $\mathcal{O}_G(\mu)$, for generic $\mu \in \phi(M)$ then $\mathcal{A} + \mathcal{B}$ is complete (commutative) algebra on M .

Now, let M be the cotangent bundle $T^*(G/H)$ with the natural G action. By ds_0^2 we shall denote the normal G -invariant Riemannian metric on the homogeneous space G/H , induced by a bi-invariant metric on G .

The Hamiltonian of the metric ds_0^2 Poisson commute with \mathcal{F}_1 and \mathcal{F}_2 . Thus, from Theorem 5.3 we get

Theorem 5.4 [15] *Let G be a compact Lie group. The geodesic flows of normal metrics ds_0^2 on the homogeneous spaces G/H are completely integrable (in non-commutative sense).*

Remark 5.1 The dimension of invariant isotropic tori T^*G/H is $\dim pr_{\mathfrak{v}}(\text{ann}_{\mathfrak{g}}(v))$ (see remark 6.1 in the next section).

Note that for \mathcal{F}_1 and \mathcal{F}_2 we can take analytic functions, polynomial in momenta. For example, in the case of Lie groups, these functions are polynomials on \mathfrak{g}^* shifted to T^*G by right and left translations, respectively.

A similar construction can be applied to bi-quotients of compact groups. Consider a subgroup U of $G \times G$ and define the action of U on G by:

$$(g_1, g_2) \cdot g = g_1 g g_2^{-1}, \quad (g_1, g_2) \in U, g \in G.$$

If the action is free then the orbit space $G//U$ is a smooth manifold called a *bi-quotient* of the Lie group G . In particular, if $U = K \times H$, where K and H are subgroups of G , then the bi-quotient $G//U$ is denoted by $K \backslash G / H$. A bi-invariant metric on G , by submersion (see section 9), induces a normal Riemannian metric ds_0^2 on $G//U$. Note that every bi-quotient $G//U$ is canonically

isometric to $\Delta G \backslash G \times G/U$, for a certain normal metric on $\Delta G \backslash G \times G/U$. Here ΔG denotes the diagonal subgroup (see [82]).

On $T^*(K \backslash G/H)$ there exist algebras \mathcal{F}_1 and \mathcal{F}_2 analogous to the above algebras on homogeneous spaces. For \mathcal{F}_1 we take all polynomials on \mathfrak{g}^* invariant with respect to the K -action and extend them to right invariant functions on T^*G . These functions give well defined functions on $T^*(K \backslash G/H)$ since they are invariant with respect to the $K \times H$ -action on T^*G . Similarly, for \mathcal{F}_2 we take all polynomials on \mathfrak{g}^* invariant with respect to the H -action, extend them to left invariant functions on T^*G and consider as functions on $T^*(K \backslash G/H)$. (When K is trivial these algebras are precisely the algebras described above). It is clear that in such a way we obtain integrals of geodesic flow of ds_0^2 .

Theorem 5.5 [16] *The algebra of functions $\mathcal{F}_1 + \mathcal{F}_2$ is complete:*

$$\text{ddim}(\mathcal{F}_1 + \mathcal{F}_2) + \text{dind}(\mathcal{F}_1 + \mathcal{F}_2) = \dim T^*(K \backslash G/H)$$

and, therefore, the geodesic flows of ds_0^2 on the bi-quotient $K \backslash G/H$ is completely integrable (in non-commutative sense).

6 Complete commutative algebras on $T^*(G/H)$

6.1 Mishchenko–Fomenko conjecture

Mishchenko and Fomenko stated the conjecture that non-commutative integrable systems $\dot{x} = X_H(x)$ are integrable in the usual commutative sense by means of integrals that belong to the same functional class as the original non-commutative algebra of integrals [55, 31]. Note that, locally, non-commutative integrability always implies commutative integrability. For example, we can just take commuting functions $\{I_1, \dots, I_r, p_1^2 + q_1^2, \dots, p_k^2 + q_k^2\}$.

When $\mathcal{F} = \text{span}_{\mathbb{R}}\{f_1, \dots, f_l\}$ is a finite-dimensional Lie algebra, this conjecture is proved for compact manifolds M by Mishchenko and Fomenko and for non-compact manifolds (under the assumption that all iso-energy levels $H^{-1}(h)$ are compact) by Brailov. Then the commuting integrals can be taken as polynomials in f_1, \dots, f_l (see [56, 21, 31]).

Recently, the conjecture has been proved in C^∞ -smooth case. This means that the r -dimensional invariant isotropic tori \mathbb{T}^r can be organized into larger, n -dimensional Lagrangian tori \mathbb{T}^n which are the level sets of the commutative algebra of smooth integrals $\{g_1, \dots, g_n\}$. The point is that local commuting integrals $\{I_1, \dots, I_r, p_1^2 + q_1^2, \dots, p_k^2 + q_k^2\}$ defined on different toroidal domains can be glued smoothly on the whole manifold [16].

If \mathcal{F} is any algebra of functions on symplectic (or Poisson) manifold, then we shall say that $\mathcal{A} \subset \mathcal{F}$ is a *complete subalgebra* if

$$\text{ddim} \mathcal{A} + \text{dind} \mathcal{A} = \text{ddim} \mathcal{F} + \text{dind} \mathcal{F}.$$

Thus, the conjecture can be stated as follows. Let \mathcal{F} be a complete algebra on a symplectic (or Poisson) manifold. Then one can find a complete commutative subalgebra \mathcal{A} of \mathcal{F} , i.e., commutative subalgebra with differential dimension

$$\text{ddim} \mathcal{A} = \frac{1}{2}(\text{ddim} \mathcal{F} + \text{dind} \mathcal{F}).$$

Here instead of \mathcal{F} we usually have to consider its functional extension $\tilde{\mathcal{F}}$, that is, the algebra formed by the functions $h = h(f_1, f_2, \dots, f_l)$, $f_i \in \mathcal{F}$, where h

is polynomial, real-analytic or C^∞ -smooth depending on the class of functions we want to work with.

6.2 Integrable pairs

We turn back to the geodesic flows on homogeneous spaces. We have shown that the non-commutative integrability implies the classical commutative integrability by means of C^∞ -smooth integrals [16]. Thus, a more delicate problem remains: the construction of complete commutative algebras of integrals of ds_0^2 that are *polynomial in momenta*.

Let \mathcal{A} be a complete commutative algebra on generic orbits in $\Phi(T^*(G/H))$ and let \mathcal{B} be a complete commutative subalgebra of \mathcal{F}_2 . Then, according to Corollary 5.1, $\mathcal{A} + \mathcal{B}$ is a complete commutative algebra on $T^*(G/H)$.

There is a well known construction, called the argument shift method [54], which allows us to obtain a complete commutative family of polynomials on every coadjoint orbit of a compact group. For regular orbits this is proved by Mishchenko and Fomenko [54]. For singular orbits there are several different proofs: by Mikityuk [57], Brailov [21] and Bolsinov [9]. (Note that $\Phi(T^*(G/H))$ can be often a subset of singular set in \mathfrak{g}^* .) Thus, to construct a complete commutative algebra of functions on $T^*(G/H)$ we need to find a complete commutative subalgebra \mathcal{B} of G -invariant functions on $T^*(G/H)$.

For symmetric and weakly symmetric spaces (spherical pairs) the algebra \mathcal{F}_2 is already commutative. In a neighborhood of a generic point $x \in T^*(G/H)$ each G -invariant function f can be expressed as $f = h \circ \Phi$ and thus we can use just functions from \mathcal{F}_1 to get the integrability of any G -invariant geodesic flow on G/H . Spherical pairs for G simple and semisimple are classified in [48] and [59], respectively.

To discuss the general case, one first needs to describe the structure of the algebra \mathcal{F}_2 . Let us fix some bi-invariant metric on G , i.e., Ad_G -invariant scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} . We can identify \mathfrak{g}^* and \mathfrak{g} by $\langle \cdot, \cdot \rangle$ and $T^*(G/H)$ and $T(G/H)$ by the corresponding normal metric ds_0^2 . Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{v}$ be the orthogonal decomposition of \mathfrak{g} , where \mathfrak{h} is the Lie algebra of H . Then G -invariant functions on $T^*(G/H)$ are in one-to-one correspondence with Ad_H invariant polynomials on \mathfrak{v} . Within this identification, the Poisson bracket on $T^*(G/H)$ corresponds to the following bracket on $\mathbb{R}[\mathfrak{v}]^H$

$$\{f(v), g(v)\}_{\mathfrak{v}} = -\langle v, [\nabla f(v), \nabla g(v)] \rangle, \quad f, g : \mathfrak{v} \rightarrow \mathbb{R}, \quad (6.1)$$

where $\mathbb{R}[\mathfrak{v}]^H$ denote the algebra of Ad_H -invariant polynomials on \mathfrak{v} (see Thimm [78]). We have

$$\text{ddim } \mathbb{R}[\mathfrak{v}]^H = \dim \mathfrak{v} - \dim \mathfrak{h} + \dim \text{ann}_{\mathfrak{h}}(v), \quad (6.2)$$

$$\text{dind } \mathbb{R}[\mathfrak{v}]^H = \dim \text{pr}_{\mathfrak{v}}(\text{ann}_{\mathfrak{g}}(v)) = \dim \text{ann}_{\mathfrak{g}}(v) - \dim \text{ann}_{\mathfrak{h}}(v) \quad (6.3)$$

for generic $v \in \mathfrak{v}$ [15]. Here $\text{ann}_{\mathfrak{g}}(v)$ and $\text{ann}_{\mathfrak{h}}(v)$ denote the annihilators of v in \mathfrak{g} and \mathfrak{h} respectively: $\text{ann}_{\mathfrak{g}}(v) = \{\eta \in \mathfrak{g}, [\eta, v] = 0\}$, $\text{ann}_{\mathfrak{h}}(v) = \{\eta \in \mathfrak{h}, [\eta, v] = 0\}$. By *genericity* of $v \in \mathfrak{v}$ we mean that the dimensions of $\text{ann}_{\mathfrak{g}}(v)$ and $\text{ann}_{\mathfrak{h}}(v)$ are minimal.

From (6.3), the condition that a commutative subalgebra $\mathcal{B} \subset \mathbb{R}[\mathfrak{v}]^H$ is complete can be rewritten in the following form:

$$\text{ddim } \mathcal{B} = \frac{1}{2}(\text{ddim } \mathbb{R}[\mathfrak{v}]^H + \text{dind } \mathbb{R}[\mathfrak{v}]^H) = \dim \mathfrak{v} - \frac{1}{2} \dim \mathcal{O}_G(v), \quad (6.4)$$

for a generic $v \in \mathfrak{v}$, where $\mathcal{O}_G(v)$ is the adjoint orbit of G .

Remark 6.1 Every Casimir function of the algebra of G -invariant functions \mathcal{F}_2 in a neighborhood of a generic point $x \in T^*(\mathfrak{G}/\mathfrak{H})$ is of the form $h \circ \Phi$, where h is a local invariant of the (co)adjoint representation. Therefore

$$\text{dind}(\mathcal{F}_1 + \mathcal{F}_2) = \text{dind} \mathcal{F}_1 = \text{dind} \mathcal{F}_2 = \text{ddim}(\mathcal{F}_1 \cap \mathcal{F}_2) = \text{dind} \mathbb{R}[\mathfrak{v}]^H.$$

In particular, the phase space of the geodesic flow of a normal metric is foliated by $\dim pr_{\mathfrak{v}}(\text{ann}_{\mathfrak{g}}(v))$ -dimensional isotropic tori.

Remark 6.2 There is a nice geometrical description of the algebra $\mathbb{R}[\mathfrak{v}]^H$. We can pass from \mathfrak{v} to the orbit space \mathfrak{v}/H , with respect to the natural adjoint action of H on \mathfrak{v} . Denote by $[v]$ the orbit of the Ad_H -action through v . It is clear that one can consider $\mathbb{R}[\mathfrak{v}]^H$ as the algebra $\mathbb{R}[\mathfrak{v}/H]$ of functions on the orbit space, with respect to $\{\cdot, \cdot\}'$ – the reduced bracket of (6.1). Note that \mathfrak{v}/H is not smooth. However, in a neighborhood of a generic point, it is a smooth manifold of dimension $(\dim \mathfrak{v} - \dim \mathfrak{h} + \dim \text{ann}_{\mathfrak{h}}(v))$ (minimality of dimensions of $\text{ann}_{\mathfrak{h}}(v)$ and $\text{ann}_{\mathfrak{g}}(v)$ means exactly that $[v]$ is a smooth point and the bracket $\{\cdot, \cdot\}'$ has maximal rank at $[v]$). Now, the completeness of \mathcal{B} as a subalgebra in \mathcal{F}_2 is equivalent to the condition that \mathcal{B} is a complete algebra on the "singular" Poisson manifold $(\mathfrak{v}/H, \{\cdot, \cdot\}')$.

Remark 6.3 The number $\frac{1}{2}(\text{ddim} \mathbb{R}[\mathfrak{v}]^H - \text{dind} \mathbb{R}[\mathfrak{v}]^H)$ is equal to the codimension $\delta(G^{\mathbb{C}}, H^{\mathbb{C}})$ of maximal dimension orbits of the Borel subgroup $B \subset G^{\mathbb{C}}$ in the complex algebraic variety $G^{\mathbb{C}}/H^{\mathbb{C}}$ (see [60]). The number $\delta(G^{\mathbb{C}}, H^{\mathbb{C}})$ is called the *complexity* of $G^{\mathbb{C}}/H^{\mathbb{C}}$ [66].

Example 6.1 Following [61, 60], we shall say that (G, H) is an *almost spherical pair* if

$$\text{ddim} \mathbb{R}[\mathfrak{v}]^H = 2 + \text{dind} \mathbb{R}[\mathfrak{v}]^H, \quad (6.5)$$

or equivalently, if the complexity of $G^{\mathbb{C}}/H^{\mathbb{C}}$ is equal to one. They are classified, for G compact and semisimple in [66, 61]. As complete commutative algebra on $T^*(G/H)$ we can take arbitrary complete commutative subalgebra $\mathcal{A} \subset \mathcal{F}_1$ and one G -invariant function functionally independent of \mathcal{F}_1 (see Mikityuk and Stepin [61]). The examples of Thimm [78] $((SO(n), SO(n-2)))$ and Paternain and Spatzier [69] $((SU(3), \mathbb{T}^2))$ are almost spherical pairs. In our notation, as a complete algebra $\mathcal{B} \subset \mathbb{R}[\mathfrak{v}]^H$ we can take all the Casimir functions and an arbitrary non Casimir function in $\mathbb{R}[\mathfrak{v}]^H$.

Definition 6.1 We will call (G, H) an *integrable pair*, if there exists a complete commutative subalgebra \mathcal{B} in $\mathbb{R}[\mathfrak{v}]^H$.

We can summarize the above considerations as follows.

Theorem 6.1 *If (G, H) is an integrable pair then the geodesic flow of a normal metric ds_0^2 is completely integrable in the commutative sense by means of analytic integrals, polynomial in velocities.*

Therefore, the Mishchenko–Fomenko conjecture for the non-commutatively integrable geodesic flow of the metric ds_0^2 on G/H can be stated as follows.

Conjecture 6.1 All pairs (G, H) are integrable.

Spherical and almost spherical pairs (G, H) are simplest examples of integrable pairs. In the section 8, following [15, 17], two natural methods for constructing commutative families of Ad_H invariant functions, namely the shift-argument method and chain of subalgebras method, are presented. In many examples (Stiefel manifolds, flag manifolds, orbits of the adjoint actions of compact Lie groups etc.) we have proved that those methods lead to complete commutative algebras.

7 Integrable deformations of normal metrics

Besides relation with the Mishchenko–Fomenko conjecture, we shall explain how, with a help of commuting integrals, one can construct new integrable geodesic flows on homogeneous spaces.

Submersion metrics. Let A be a complete commutative algebra on a generic orbit in $\Phi(T^*(G/H))$. Then, according to Corollary 5.1, $\mathcal{A} + \mathcal{F}_2$ is a complete algebra on $T^*(G/H)$. Let $h_C(\xi) = \frac{1}{2}\langle C(\xi), \xi \rangle$ be a quadratic positive definite polynomial in A . Then $h_C \circ \Phi$ is the Hamiltonian of the geodesic flow of a certain metric that we shall denote by ds_C^2 . The metric ds_C^2 has the following nice geometrical meaning. This is the submersion metric of the right-invariant Riemannian metric on G whose Hamiltonian function is obtained from $h_C(\xi)$ by right translations. The geodesic flow of ds_C^2 is completely integrable since $h_C \circ \Phi$ commutes with every function from $\mathcal{A} + \mathcal{F}_2$. The dimension of invariant tori is equal to

$$\text{dind}(\mathcal{A} + \mathcal{F}_2) = \dim \mathcal{O}_G(v) + \dim pr_{\mathfrak{v}}(\text{ann}_{\mathfrak{g}}(v)),$$

for a generic $v \in \mathfrak{v}$ (see [15]).

Notice that the argument shift method [54] (see below) always allows us to construct a commutative subalgebra $\mathcal{A} \subset \mathcal{F}_1$ which contains non-trivial quadratic functions. Thus some integrable (in non-commutative sense) deformations of ds_0^2 always exist.

This construction for symmetric spaces is done by Brailov [21, 22].

G-invariant metrics. Let \mathcal{B} be a complete commutative subalgebra of \mathcal{F}_2 . Let $h_G \in \mathcal{B}$ be a G -invariant function, quadratic in momenta and positive definite. Then h_G can be considered as the Hamiltonian of the geodesic flow of a certain G -invariant metric ds_G^2 . The geodesic flow of ds_G^2 is completely integrable since it admits the complete algebra of first integrals $\mathcal{F}_1 + \mathcal{B}$ (see Corollary 5.1). The dimension of invariant tori is equal to

$$\text{dind}(\mathcal{F}_1 + \mathcal{B}) = \text{dind} \mathcal{B} = \dim \mathfrak{v} - \frac{1}{2} \dim \mathcal{O}_G(v),$$

for a generic $v \in \mathfrak{v}$.

Non-invariant metrics. Let $h_C \circ \Phi$ and h_G be as above and let $H_{\lambda_1, \lambda_2} = \lambda_1 h_C \circ \Phi + \lambda_2 h_G$ be positive definite. Then H_{λ_1, λ_2} is the Hamiltonian of the metric which we shall denote by $ds_{\lambda_1, \lambda_2}^2$. The family of metrics $ds_{\lambda_1, \lambda_2}^2$ has completely integrable geodesic flows, but with no obvious symmetries. The phase space $T^*(G/H)$ is then foliated by invariant Lagrangian tori that are level sets of the complete commutative algebra of functions $\mathcal{A} + \mathcal{B}$.

Example 7.1 The most natural and simplest example of integrable deformations is as follows. Consider a compact Lie group G as a homogeneous space. The geodesic flow of the biinvariant metric ds_0^2 is completely integrable in non-commutative sense and the dimension of invariant isotropic tori in T^*G is equal to $\text{ind}G$. The first integrals of the flow are all left- and right-invariant functions on $T^*G = TG$. To obtain integrable deformations of ds_0^2 one should consider a complete commutative subalgebra $\mathcal{B} \subset \mathcal{F}_2$, where \mathcal{F}_2 is the algebra of left-invariant functions. Such a subalgebra can be constructed by the argument shift. Moreover one can describe all quadratic functions in \mathcal{B} (see [54]) in the following way. Consider the linear operator $C_{a,b,D} : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by

$$C_{a,b,D}(x) = \text{ad}_a^{-1} \text{ad}_b(x_1) + D(x_2),$$

where $x = x_1 + x_2$, $x_2 \in \mathfrak{k}$, $x_1 \in \mathfrak{k}^\perp$, $\mathfrak{k} \subset \mathfrak{g}$ is a Cartan subalgebra, $a, b \in \mathfrak{k}$ and a is regular, $D : \mathfrak{k} \rightarrow \mathfrak{k}$ is an arbitrary operator. Then we consider the quadratic form $\frac{1}{2}\langle C_{a,b,D}(x), x \rangle$ on \mathfrak{g} and extend it to the whole tangent bundle TG by left translations. As a result we obtain a quadratic function $h_{a,b,D}^{\text{left}}$ which can be considered as the Hamiltonian of a left invariant metric on G . Its geodesic flow will be integrable and the algebra of integrals consists of two parts: \mathcal{B} (commutative part) and \mathcal{F}_1 (non-commutative part which consists of all right-invariant functions). For such flows, the dimension of invariant isotropic tori in $T^* = TG$ will be equal to $\frac{1}{2}(\dim G + \text{ind } G)$. But we can continue this deformation procedure by choosing a complete commutative subalgebra $\mathcal{A} \subset \mathcal{F}_1$. This can be done just in the same way as for \mathcal{F}_2 . As a result we shall obtain a right-invariant quadratic function $h_{a',b',D'}^{\text{right}}$ which also gives an integrable geodesic flows on G . But now we can take the sum $h_{a,b,D}^{\text{left}} + h_{a',b',D'}^{\text{right}}$ which also gives an integrable geodesic flow whose algebra of integrals $\mathcal{A} + \mathcal{B}$ is commutative and, consequently, invariant tori are Lagrangian, i.e., of dimension $\dim G$.

8 Methods and examples

8.1 Argument shift method

Let $\mathbb{R}[\mathfrak{g}]^G$ be the algebra of Ad_G -invariant polynomials on \mathfrak{g} . Mishchenko and Fomenko showed that the polynomials

$$A_a = \{p_a^\lambda = p(\cdot + \lambda a), \lambda \in \mathbb{R}, p \in \mathbb{R}[\mathfrak{g}]^G\}$$

obtained from the invariants by shifting the argument are all in involution [54]. Furthermore, for *every* adjoint orbit in \mathfrak{g} , one can find $a \in \mathfrak{g}$, such that A_a is a complete involutive set of functions on this orbit. For regular orbits it is proved by Mishchenko and Fomenko [54]. For singular orbits there are several different proofs by Mikityuk [57], Brailov [21] and Bolsinov [9].

Thus, as was already mentioned, the argument shift method allows us to construct a complete commutative subalgebra in \mathcal{F}_1 . Now we want to use it to construct such a subalgebra in \mathcal{F}_2 .

By \mathcal{B}_a denote the restriction of A_a to \mathfrak{v} :

$$\mathcal{B}_a = \{p_a^\lambda(v) = p(v + \lambda a), \lambda \in \mathbb{R}, p \in \mathbb{R}[\mathfrak{g}]^G, v \in \mathfrak{v}\}$$

It can be easily checked that if H is the subgroup of the isotropy group G_a :

$$H \subset G_a = \{g \in G, \text{Ad}_g(a) = a\},$$

(i.e., $\mathfrak{h} \subset \text{ann}_{\mathfrak{g}}(a)$) for some $a \in \mathfrak{g}$ then \mathcal{B}_a will be an algebra of Ad_H -invariant polynomials. Also, we have the following simple observation. If f_1 and f_2 are in involution $\{f_1, f_2\}_{\mathfrak{g}} = 0$ and their restrictions to \mathfrak{v} : $p_1 = f_1|_{\mathfrak{v}}$, $p_2 = f_2|_{\mathfrak{v}}$ are Ad_H -invariant; then $\{p_1, p_2\}_{\mathfrak{v}} = 0$. Thus, \mathcal{B}_a is a commutative subalgebra of $\mathbb{R}[\mathfrak{v}]^H$.

In order to estimate the number of independent functions, obtained by shifting the argument, we look at the algebra \mathcal{B}_a from the point of view of the bi-Hamiltonian system theory. This approach gives us a possibility to use, in particular, the completeness criterion proved by the first author in [9].

Let $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ be compatible Poisson structures on a manifold N . Compatibility means that any linear combination of $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ with constant coefficients is again a Poisson structure. So we have a family of Poisson structures:

$$\Lambda = \{\lambda_1 \{\cdot, \cdot\}_1 + \lambda_2 \{\cdot, \cdot\}_2, \lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1^2 + \lambda_2^2 \neq 0\}.$$

By r denote the rank of a generic bracket in Λ . For each bracket $\{\cdot, \cdot\} \in \Lambda$ of rank r , we consider the set of its Casimir functions. Let \mathcal{F}_{Λ} be the union of these sets. Then \mathcal{F}_{Λ} is an involutive set with respect to every Poisson bracket from Λ . Together with Λ , consider its natural complexification $\Lambda^{\mathbb{C}} = \{\lambda_1 \{\cdot, \cdot\}_1 + \lambda_2 \{\cdot, \cdot\}_2, \lambda_1, \lambda_2 \in \mathbb{C}, |\lambda_1|^2 + |\lambda_2|^2 \neq 0\}$. Here, for $\{\cdot, \cdot\} \in \Lambda^{\mathbb{C}}$, we consider $\{\cdot, \cdot\}(x)$ as a complex valued skew-symmetric bilinear form on the complexification of the co-tangent space $(T_x^*N)^{\mathbb{C}}$.

Theorem 8.1 (Bolsinov, 1989 [9]) *Let $\{\cdot, \cdot\} \in \Lambda$ and $\text{rank } \{\cdot, \cdot\}(x) = r$. Then \mathcal{F}_{Λ} is complete at x with respect to $\{\cdot, \cdot\}$ if and only if $\text{rank } \{\cdot, \cdot\}'(x) = r$ for all $\{\cdot, \cdot\}' \in \Lambda^{\mathbb{C}}$.*

Note that if \mathcal{F}_{Λ} is complete at x then it is complete in some neighborhood U of x , or, if functions \mathcal{F}_{Λ} are analytic, in N .

Now, let us pass from \mathfrak{v} to the orbit space \mathfrak{v}/H (see remark 6.2). It is easy to see that the algebra $\mathbb{R}[\mathfrak{v}]^H$ is closed with respect to the a -bracket defined by

$$\{f(x), g(x)\}_a = \langle a, [\nabla g(x), \nabla f(x)] \rangle.$$

That is why $\{\cdot, \cdot\}_a$ induces a Poisson bracket $\{\cdot, \cdot\}'_a$ on the orbit space.

Moreover, it is well known that the Lie-Poisson brackets and a -brackets are compatible on \mathfrak{g} [9, 10]. Thus, the brackets $\{\cdot, \cdot\}'$ and $\{\cdot, \cdot\}'_a$ are also compatible. Notice that $p(\cdot + \lambda a)$, where p is an Ad_G -invariant, is a Casimir function of the Poisson bracket $\{\cdot, \cdot\}' + \lambda \{\cdot, \cdot\}'_a$. Using the above criterion (Theorem 8.1) we have found the following conditions sufficient for \mathcal{B}_a to be complete.

Theorem 8.2 [17] *Let $H = G_a$ for some $a \in \mathfrak{g}$. Suppose that there exists generic $v \in \mathfrak{v}^{\mathbb{C}}$ such that:*

$$(C1) \quad \dim \text{ann}_{\mathfrak{g}^{\mathbb{C}}}(v + \lambda a) = \dim \text{ann}_{\mathfrak{g}^{\mathbb{C}}}(v), \quad \text{for all } \lambda \in \mathbb{C},$$

$$(C2) \quad \dim \text{ann}_{\mathfrak{h}^{\mathbb{C}} p r_{\mathfrak{h}^{\mathbb{C}}}([\text{ad}_a^{-1} v, v])} = \dim \text{ann}_{\mathfrak{g}^{\mathbb{C}}}(v).$$

Then \mathcal{B}_a is a complete commutative subalgebra in $\mathbb{R}[\mathfrak{v}]^H$. In particular (G, G_a) is an integrable pair.

The homogeneous space G/G_a is the adjoint orbit $\mathcal{O}_G(a)$ of the G -action on \mathfrak{g} . If a is regular in \mathfrak{g} then G_a is a maximal torus and G/H is usually called a *flag manifold*. In this simplest case conditions (C1) and (C2) can be easily verified.

Corollary 8.1 (Bordemann [19], see also [15]) *Let H be a maximal torus in a compact connected Lie group G . If $a \in \mathfrak{h}$ is regular, then \mathcal{B}_a is a complete commutative subalgebra in $\mathbb{R}[\mathfrak{v}]^H$. In particular (G, H) is an integrable pair.*

We think that conditions (C1) and (C2) hold for all compact Lie groups and for each $a \in \mathfrak{g}$. In [17] we have verified them for $U(n)$ ($SU(n)$), and then (joint work with E. Buldaeva [23]) for $SO(n)$ and $Sp(n)$.

Theorem 8.3 *Let G be a classical compact simple Lie group ($SO(n)$, $SU(n)$ or $Sp(n)$) and $\mathcal{O}_G(a)$ be an arbitrary adjoint orbit of G . Then (G, G_a) is an integrable pair and \mathcal{B}_a is a complete commutative subalgebra in $\mathbb{R}[\mathfrak{v}]^H$.*

For example, if we take $G = U(n)$ and $a \in \mathfrak{u}(n)$ of the form

$$a = \text{diag}(a_1, \dots, a_n) = \text{diag}(\underbrace{\alpha_1, \dots, \alpha_1}_{k_1}, \underbrace{\alpha_2, \dots, \alpha_2}_{k_2}, \dots, \underbrace{\alpha_r, \dots, \alpha_r}_{k_r})$$

then $\mathcal{O}_{U(n)}(a) = U(n)/U(k_1) \times \dots \times U(k_r)$ and $(U(n), U(k_1) \times \dots \times U(k_r))$ is an integrable pair.

Pairs $(U(n), U(k_1) \times \dots \times U(k_r))$ are integrable for $k_1 + \dots + k_r < n$ as well. Indeed, there are commutative subalgebra $\mathfrak{t} \subset \mathfrak{v}$ and diagonal matrix a such that $\mathfrak{u}(k_1) + \dots + \mathfrak{u}(k_r) + \mathfrak{t} = \text{ann}_{\mathfrak{u}(n)}(a)$. Let $\mathbb{R}[\mathfrak{t}]$ be the algebra of polynomials on \mathfrak{t} considered as functions on \mathfrak{v} . Simple calculations show that $\mathcal{B}_a + \mathbb{R}[\mathfrak{t}]$ is a complete commutative algebra of $\text{Ad}_{U(k_1) \times \dots \times U(k_r)}$ invariants.

Just in the same way we can prove that if the pair (G, G_a) is integrable and if $H \subset G_a$ is a normal subgroup such that G_a/H is commutative, then (G, H) is an integrable pair as well.

8.2 Chains of subalgebras

Trofimov and Thimm devised a method for constructing functions in involution on a Lie algebra \mathfrak{g} by using chains of subalgebras [80, 78]. The idea is very simple, but important. The invariant polynomials commute with all functions on \mathfrak{g} . If we have some subalgebra $\mathfrak{g}_1 \subset \mathfrak{g}$, then invariant polynomials on \mathfrak{g}_1 (naturally extended to the whole \mathfrak{g}) commute between themselves, but also with invariant polynomials on \mathfrak{g} . By induction, we come to the following construction.

Suppose we are given a chain of connected compact subgroups $G_1 \subset G_2 \subset \dots \subset G_n = G$, and the corresponding chain of subalgebras in \mathfrak{g} :

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \dots \subset \mathfrak{g}_n = \mathfrak{g} \quad (8.1)$$

Let A_i be the algebra of invariants on \mathfrak{g}_i considered as a subalgebra in $\mathbb{R}[\mathfrak{g}]$. Then $A_1 + \dots + A_n$ is a commutative subalgebra of polynomials on \mathfrak{g} [80, 78].

Example 8.1 The natural filtrations $\mathfrak{so}(2) \subset \mathfrak{so}(3) \subset \dots \subset \mathfrak{so}(n)$ and $\mathfrak{u}(1) \subset \mathfrak{u}(2) \subset \dots \subset \mathfrak{u}(n)$ lead to complete commutative algebras on $\mathfrak{so}(n)$ and $\mathfrak{u}(n)$ respectively (see Thimm [78]).

Now, if we have $H \subset G_1$ then polynomials in $A_1 + \dots + A_n$ are Ad_H -invariant. Therefore $\mathcal{B} = \mathcal{B}_1 + \dots + \mathcal{B}_n$ is a commutative subalgebra of $\mathbb{R}[\mathfrak{v}]^H$, where \mathcal{B}_i is restriction of A_i to \mathfrak{v} .

With respect to (8.1) we have the orthogonal decomposition

$$\mathfrak{v} = \mathfrak{v}_1 + \mathfrak{v}_2 + \dots + \mathfrak{v}_n,$$

such that $\mathfrak{g}_i = \mathfrak{h} + \mathfrak{v}_1 + \dots + \mathfrak{v}_i$. Let us define r_1, \dots, r_n by:

$$r_i = \max_{v \in \mathfrak{v}_1 + \dots + \mathfrak{v}_i} \dim pr_{\mathfrak{v}_i} \text{span} \{ \nabla p(v), p \in \mathbb{R}[\mathfrak{g}_i]^{G_i} \}.$$

Let $R = r_1 + \dots + r_n$. It is clear that the number of independent functions in \mathcal{B} is greater or equal to R . Hence, if $R = \dim \mathfrak{v} - \frac{1}{2} \dim \mathcal{O}_G(v)$ then \mathcal{B} is complete. An algorithm for computing numbers r_i is given by Bazaikin [7].

In the next theorem we give examples of some integrable pairs with complete algebras $\mathcal{B} \subset \mathbb{R}[\mathfrak{v}]^H$ obtained by using chains of subalgebras.

Theorem 8.4 [17]

$$\begin{array}{ll} (SO(n), SO(k_1) \times SO(k_2)), & k_1, k_2 \geq 0, k_1 + k_2 \leq n \\ (U(n), U(1)^{k_1} \times U(k_2) \times U(k_3)), & k_1, k_2, k_3 \geq 0, k_1 + k_2 + k_3 \leq n \\ (U(n), SO(k)), & k \leq n \\ (SO(n_1) \times SO(n_2), SO(k)), & k \leq n_1, n_2 \\ (U(n_1) \times U(n_2), U(k)), & k \leq n_1, n_2 \end{array}$$

are integrable pairs. In the last two examples $SO(k)$ and $U(k)$ are diagonally embedded into $SO(n_1) \times SO(n_2)$ and $U(n_1) \times U(n_2)$ respectively.

Example 8.2 As an example we shall indicate the chain for Stiefel manifold $SO(m)/SO(k)$ (see [15]):

$$\begin{aligned} \mathfrak{g}_1 &= \mathfrak{so}(k+1) \subset \mathfrak{g}_2 = \mathfrak{so}(k+2) \subset \dots \subset \mathfrak{g}_{m-k} \subset \mathfrak{so}(m), \\ r_i &= i, \quad i = 1, \dots, k, \\ r_i &= \left\lfloor \frac{k+i}{2} \right\rfloor, \quad i = k+1, k+2, \dots, m-k. \end{aligned}$$

Example 8.3 In Theorem 8.4, we consider naturally embedded subgroups (as block matrices). However, the same construction can be applied to some other embeddings. As an example, let us consider the so-called Aloff–Wallach spaces $M_{k,l} = SU(3)/T_{k,l}$ [3], where

$$T_{k,l} = \left\{ \begin{pmatrix} e^{2\pi i k \theta} & 0 & 0 \\ 0 & e^{2\pi i l \theta} & 0 \\ 0 & 0 & e^{-2\pi i (k+l) \theta} \end{pmatrix}, \theta \in \mathbb{R} \right\}, \quad k, l \in \mathbb{Z}, |k| + |l| \neq 0, kl > 0.$$

Among the spaces $M_{k,l}$ there are infinitely many with different cohomological structures: if k, l are relatively prime, then $H^4(M_{k,l}, \mathbb{Z}) = \mathbb{Z}/r\mathbb{Z}$, with $r = |k^2 + l^2 + kl|$ [3].

Consider the following chain of subgroups $G_0 = T_{k,l} \subset G_1 \subset G_2 = SU(3)$, where

$$G_1 = \left\{ \begin{pmatrix} g & 0 \\ 0 & \det g^{-1} \end{pmatrix} \in SU(3), g \in U(2) \right\} \cong U(2).$$

Let $\mathfrak{v} = \mathfrak{v}_1 + \mathfrak{v}_2$ be the orthogonal decomposition as above

$$\begin{aligned} \mathfrak{v}_1 &= \left\{ \begin{pmatrix} ia & z_{12} & 0 \\ -\bar{z}_{12} & ib & 0 \\ 0 & 0 & -i(a+b) \end{pmatrix}, ka + lb + (k+l)(a+b) = 0 \right\}, \\ \mathfrak{v}_2 &= \left\{ \begin{pmatrix} 0 & 0 & z_{13} \\ 0 & 0 & z_{23} \\ -\bar{z}_{13} & -\bar{z}_{23} & 0 \end{pmatrix} \right\}. \end{aligned}$$

The algebra of functions $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$ is complete. Indeed, \mathcal{B}_1 is generated by $f_1 = a + b$ and $f_2 = \langle v_1, v_1 \rangle$ ($r_1 = 2$), \mathcal{B}_2 is generated by $f_3 = \langle v_1 + v_2, v_1 + v_2 \rangle$ and $f_4 = \text{tr}(v_1 + v_2)^3$ ($r_2 = 2$) and $2 + 2 = 4 = \dim \mathfrak{v} - \frac{1}{2}\mathcal{O}_{SU(3)}(v) = 7 - 3$.

Aloff and Wallach proved that the $SU(3)$ -invariant Riemannian metrics ds_t^2 on $M_{k,l}$ obtained from the quadratic forms

$$B_t(v, v) = (1 + t)\langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle, \quad v = v_1 + v_2, \quad v_i \in \mathfrak{v}_i, \quad -1 < t < 0$$

have positive sectional curvature. Since the Hamiltonian functions of metrics ds_t^2 belong to \mathcal{B} , the geodesic flows of ds_t^2 are completely integrable.

8.3 Generalized chain method

Let θ be a Cartan involution on \mathfrak{g} and let $\mathfrak{g} = \mathfrak{l} + \mathfrak{m}$ be the corresponding orthogonal decomposition into the eigen-spaces of θ . Then $(\mathfrak{g}, \mathfrak{l})$ is called a *symmetric pair* and the following relations hold:

$$[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}, \quad [\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{l}.$$

Consider the following algebra of polynomials on \mathfrak{g} :

$$A_{\mathfrak{g}, \mathfrak{l}} = \{f(x) = p(\lambda l + w), \quad p \in \mathbb{R}[\mathfrak{g}]^G, \quad \lambda \in \mathbb{R}\}, \quad (8.2)$$

where $x = l + w$ is the orthogonal decomposition ($l \in \mathfrak{l}, w \in \mathfrak{m}$). Let $\mathbb{R}[\mathfrak{l}]$ be the algebra of polynomials on \mathfrak{l} considered as a subalgebra in $\mathbb{R}[\mathfrak{g}]$.

Theorem 8.5 (i) $A_{\mathfrak{g}, \mathfrak{l}}$ is a commutative algebra of polynomials in involution with polynomials from $\mathbb{R}[\mathfrak{l}]$, i.e., commutative subalgebra in $\mathbb{R}[\mathfrak{g}]^L$.

(ii) $A_{\mathfrak{g}, \mathfrak{l}} + \mathbb{R}[\mathfrak{l}]$ is a complete algebra of polynomials on \mathfrak{g} . In particular, if $A_{\mathfrak{l}}$ is a complete commutative subalgebra of $\mathbb{R}[\mathfrak{l}]$, then $A_{\mathfrak{g}, \mathfrak{l}} + A_{\mathfrak{l}}$ will be a complete commutative algebra on \mathfrak{g} .

This result was first proof by Mikityuk [58]. Also, Theorem 8.5 is a special case of theorem 1.5 [9]. It is related to the compatibility of the Lie-Poisson bracket $\{\cdot, \cdot\}_{\mathfrak{g}}$ and the θ -bracket defined by: $\{f, g\}_{\theta}(x) = \langle x, [\nabla g(x), \nabla f(x)]_{\theta} \rangle$, where $[\cdot, \cdot]_{\theta}$ is a new operation on \mathfrak{g} which differs from the standard one $[\cdot, \cdot]$ by the only property that \mathfrak{m} is assumed to be commutative $[l_1 + w_1, l_2 + w_2]_{\theta} = [l_1, l_2] + [l_1, w_2] + [w_1, l_2]$ (for more details and related references see [9, 10]).

Suppose we are given a chain of connected subgroups $G_1 \subset \cdots \subset G_n$ and the corresponding chain of subalgebras $\mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n$. Furthermore, suppose that either $(\mathfrak{g}_i, \mathfrak{g}_{i-1})$ is a symmetric pair or \mathfrak{v}_i is a subalgebra of \mathfrak{g}_i , for all $i = 2, 3, \dots, n$. Here $\mathfrak{g}_i = \mathfrak{g}_{i-1} + \mathfrak{v}_i$, $i = 2, 3, \dots, n$.

Let A_1 be an arbitrary complete involutive set of polynomials on \mathfrak{g}_1 . For symmetric pairs $(\mathfrak{g}_i, \mathfrak{g}_{i-1})$, let A_i be the algebra $A_{\mathfrak{g}_i, \mathfrak{g}_{i-1}}$ considered on \mathfrak{g} . Otherwise, if \mathfrak{v}_i is a subalgebra, then let A_i be an arbitrary complete involutive set of polynomials on \mathfrak{g}_i lifted to \mathfrak{g}_n (it is clear that in this case A_i commute with $A_1 + \cdots + A_{i-1}$). By induction, using Theorem 8.5, we get that $A_1 + A_1 + \cdots + A_n$ is a complete commutative algebra of polynomials on \mathfrak{g}_n .

Now, if we want to apply the above construction to the our problem, we have to prove "an inductive step", analogous to Theorem 8.5. Suppose we are given an integrable pair (L, H) . Let L be a subgroup of G such that $(\mathfrak{g}, \mathfrak{l})$ is a symmetric pair. Let

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{v}, \quad \mathfrak{l} = \mathfrak{h} + \mathfrak{v}_1, \quad \mathfrak{v} = \mathfrak{v}_1 + \mathfrak{v}_2$$

be the orthogonal decompositions. Let \mathcal{B}_1 be a complete commutative algebra of Ad_H -invariants on \mathfrak{v}_1 lifted to \mathfrak{v} and \mathcal{B}_2 be the restriction of algebra $A_{\mathfrak{g},\mathfrak{l}}$ to \mathfrak{v} :

$$\mathcal{B}_2 = \{f(v_1 + v_2) = p(\lambda v_1 + v_2), p \in \mathbb{R}[\mathfrak{g}]^G, \lambda \in \mathbb{R}\}, \quad v_1 \in \mathfrak{v}_1, \quad v_2 \in \mathfrak{v}_2.$$

From Theorem 8.5 we get that $\mathcal{B}_1 + \mathcal{B}_2$ is a commutative subalgebra of $\mathbb{R}[\mathfrak{v}]^H$.

Question 8.1 *Is $\mathcal{B}_1 + \mathcal{B}_2$ complete in $\mathbb{R}[\mathfrak{v}]^H$?*

The following particular result holds.

Theorem 8.6 [17] *Suppose that (L, H) is an integrable pair, G/L is a maximal rank symmetric space and a generic $v_1 \in \mathfrak{v}_1$ is a regular element of \mathfrak{l} . Then (G, H) is an integrable pair and $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$ is a complete commutative subalgebra of $\mathbb{R}[\mathfrak{v}]^H$.*

The conditions of Theorem 8.6 are not necessary conditions (for example, consider $(H, L, G) = (SO(n), SO(n+1), SO(n+2))$). It would be interesting to prove that algebra \mathcal{B} is always complete. Then, if we have a chain of subgroups as above, such that H is a subgroup of G_1 and that (G_1, H) is an integrable pair, we would get that (G_n, H) is an integrable pair as well. Simply, in this case we can take complete algebra $\mathcal{B}_1 + \dots + \mathcal{B}_n$, where \mathcal{B}_i is the restriction of A_i to \mathfrak{v} , for $i = 2, 3, \dots, n$, and \mathcal{B}_1 is a complete commutative algebra for (G_1, H) considered as a subalgebra of $\mathbb{R}[\mathfrak{v}]^H$.

Example 8.4 We can use Theorem 8.6 for a maximal rank symmetric space $Sp(n)/U(n)$ and integrable pairs $(U(n), U(k_1) \times \dots \times U(k_r))$ with $k_i \leq [(n+1)/2]$, $i = 1, \dots, r$ (this guaranties the regularity conditions from the theorem) to obtain the integrability of pairs $(Sp(n), U(k_1) \times \dots \times U(k_r))$.

Example 8.5 Suppose that $(SO(n_1), H_1)$ and $(SO(n_2), H_2)$ are integrable pairs, where $H_1 = SO(k_1) \times \dots \times SO(k_{r_1})$ and $H_2 = SO(l_1) \times \dots \times SO(l_{r_2})$. Suppose that generic $u_1 \in \mathfrak{u}_1$ and $u_2 \in \mathfrak{u}_2$ are regular in $\mathfrak{so}(n_1)$ and $\mathfrak{so}(n_2)$ respectively. Here \mathfrak{u}_i is the orthogonal complement of \mathfrak{h}_i in $\mathfrak{so}(n_i)$, $i = 1, 2$. (For example we can take integrable pairs $(SO(2k+n), SO(k))$ and $(SO(2l+m), SO(2l))$ from Theorem 8.4) Then, if $SO(n_1 + n_2)/SO(n_1) \times SO(n_2)$ is a maximal rank symmetric space ($n_1 = n_2 \pm 0, 1$), $(SO(n_1 + n_2), SO(k_1) \times \dots \times SO(k_{r_1}) \times SO(l_1) \times \dots \times SO(l_{r_2}))$ is an integrable pair.

Example 8.6 Take an integrable pair $(SO(n), SO(k_1) \times \dots \times SO(k_r))$ obtained by the above construction. Then, since $SU(n)/SO(n)$ is a maximal rank symmetric space, $(SU(n), SO(k_1) \times \dots \times SO(k_r))$ is an integrable pair as well.

9 Integrability and reduction

Submersions. Suppose we are given a compact Riemannian manifold (Q, g) with a completely integrable geodesic flow. Let G be a compact connected Lie group acting freely on Q by isometries. The natural question arises: will the geodesic flow on Q/G equipped with the submersion metric be integrable?

Let $\mathcal{G}_x + \mathcal{H}_x = T_x Q$ be the orthogonal decomposition of $T_x Q$, where \mathcal{G}_x is the tangent space to the fiber $G \cdot x$. By definition, the submersion metric g_{sub} is given by

$$\langle \xi_1, \xi_2 \rangle_{\rho(x)} = \langle \bar{\xi}_1, \bar{\xi}_2 \rangle_x, \quad \xi_i \in T_{\rho(x)}(Q/G), \quad \bar{\xi}_i \in \mathcal{H}_x, \quad \xi_i = d\rho(\bar{\xi}_i),$$

where $\rho : Q \rightarrow Q/G$ is the canonical projection. The vectors in \mathcal{G}_x and \mathcal{H}_x are called *vertical* and *horizontal* respectively.

Let $\Phi : T^*Q \rightarrow \mathfrak{g}^*$ be the moment map of the natural Hamiltonian G -action on T^*Q . It is well known that the reduced symplectic space $(T^*Q)_0 = \Phi^{-1}(0)/G$ is symplectomorphic to $T^*(Q/G)$. If H is the Hamiltonian function of the geodesic flow on Q then $H|_{\Phi^{-1}(0)}$ considered on the reduced space will be the Hamiltonian of the geodesic flow for the submersion metric. If we identify T^*Q and TQ by the metric g , then $\Phi^{-1}(0)$ will be the set of all horizontal vectors \mathcal{H} . Moreover, if f and g are G -invariant and $\{f, g\}|_{\mathcal{H}} = 0$ then f and g descend to Poisson commuting functions on the reduced space (see [69]). Thus, the base space of the submersion has completely integrable geodesic flow if enough G -invariant commuting functions descend to independent functions.

Paternain and Spatzier proved that if the manifold Q has geodesic flow integrable by means of S^1 -invariant integrals and if N is a surface of revolution, then the submersion geodesic flow on $Q \times_{S^1} N = (Q \times N)/S^1$ will be completely integrable [69]. The connected sum $\mathbb{CP}^{n\sharp} - \mathbb{CP}^n$ is an example (by a different method they also constructed integrable geodesic flows on $\mathbb{CP}^{2n+1\sharp} \mathbb{CP}^{2n+1}$).

Bi-quotients. Combining submersions and chain of subalgebras method, Paternain and Spatzier [69] and Bazaikin [7] proved integrability of geodesic flows of normal metrics on certain interesting bi-quotients of Lie groups. The idea is as follows.

Consider a bi-quotient $G//U$ endowed with the normal metric ds_0^2 . Let

$$\phi : TG \rightarrow \mathfrak{g} \oplus \mathfrak{g}$$

be the moment map of the $G \times G$ action on G : $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$, $(g_1, g_2) \in G \times G$, $g \in G$. Here we identified dual spaces by the bi-invariant metric on G . Let $\mathfrak{u} \subset \mathfrak{g} \oplus \mathfrak{g}$ be the Lie algebra of U and let $\mathfrak{w} = \mathfrak{u}^\perp$ be its orthogonal complement. Also, let $\mathcal{U} \subset \mathfrak{g} \cong T_e G$ be the vertical space at the neutral element of the group.

Then the horizontal space \mathcal{H} at the neutral element is the orthogonal complement of \mathcal{U} in \mathfrak{g} . Let $\xi \in \mathcal{H}$ be the horizontal vector in the neutral of the group. Then $d\phi(T_\xi \mathcal{H})$ is the vector subspace of $\mathfrak{g} \oplus \mathfrak{g}$ equal to:

$$d\phi(T_\xi \mathcal{H}) = \mathfrak{k}_\xi = (\Delta \text{ann}_{\mathfrak{g}}(\xi))^\perp \cap \mathfrak{w}, \quad (9.1)$$

where Δ denotes the diagonal embedding (see [69, 7]).

Suppose that $F = \{h_1, \dots, h_r\}$ is a commutative family of Ad_U -invariant polynomials (with respect to the Lie-Poisson bracket) on $\mathfrak{g} \oplus \mathfrak{g}$. Then $\mathcal{F} = \{\phi \circ h_1, \dots, \phi \circ h_r\}$ will be a commutative family of U -invariant functions on TG . Since we deal with analytic functions, it is clear from (9.1) that the number of independent functions on the reduced space is greater or equal to the number

$$\dim pr_{\mathfrak{k}_\xi} \text{span} \{\nabla h_1(\xi), \dots, \nabla h_r(\xi)\}, \quad \xi \in \mathcal{H}. \quad (9.2)$$

Bazaikin developed the method of computing numbers (9.2) for the families of commuting Ad_U -invariant polynomials obtained by using chain of subalgebras [7]. In particular, he proved the complete integrability of the geodesic flows of normal metrics on the 7-dimensional and 13-dimensional bi-quotients of groups $G = U(3) \times U(2) \times U(1)$ $G = U(5) \times U(4) \times U(1)$ with strictly positive sectional curvature [7].

General approach. Now we will show how some of these results, as well as some of the mentioned results on the integrability of geodesic flows on homogeneous spaces, can be obtained directly, considering the relationships between symplectic reductions and the integrability of Hamiltonian systems.

Let G be a compact connected Lie group with a free Hamiltonian action on a symplectic manifold (M, ω) . Let $\Phi : M \rightarrow \mathfrak{g}^*$ be the corresponding equivariant moment map. Let G_η be the coadjoint action isotropy group of $\eta \in \Phi(M) \subset \mathfrak{g}^*$. By (M_η, ω_η) we denote the reduced symplectic space

$$M_\eta = \Phi^{-1}(\eta)/G_\eta, \quad \omega_\eta(d\pi(\xi_1), d\pi(\xi_2)) = \omega(\xi_1, \xi_2), \quad \xi_1, \xi_2 \in T_x \Phi^{-1}(\eta),$$

where $\pi : \Phi^{-1}(\eta) \rightarrow M_\eta$ is the natural projection.

Suppose we are given an integrable G -invariant Hamiltonian system $\dot{x} = X_H(x)$ with compact iso-energy levels $M_h = H^{-1}(h)$. Then M is foliated by invariant tori in an open dense set that we shall denote by $\text{reg } M$.

The following theorem is recently proved in [38] and in a slightly different version, independently, in [87].

Theorem 9.1 *If $\text{reg } M$ intersects the submanifold $\Phi^{-1}(\eta)$ in a dense set then the reduced Hamiltonian system on (M_η, ω_η) will be completely integrable.*

Note that we do not suppose that integrals of the original system are G -invariant. Also, the general construction used in the proof of the theorem leads to smooth commuting integrals on M_η . In particular, if the Hamiltonian system $\dot{x} = X_H(x)$ is completely integrable by means of G -invariant first integrals, then G is a torus and we can use original integrals to prove the integrability of the reduced system [38].

Remark 9.1 It is obvious that reductions by a discrete group Σ of symmetries have no influence on the integrability: quasi-periodic motions on M go to the quasi-periodic motions on M/Σ . Namely, suppose that the trajectory γ fill up densely an r -dimensional torus $\mathbb{T}^r \subset M$. Then $\pi(\gamma) \subset M/\Sigma$ will fill up densely a torus of the same dimension: $\Sigma(\mathbb{T}^r)/\Sigma$.

Therefore we get

Theorem 9.2 [38] *Let a compact connected Lie group G act freely by isometries on a compact Riemannian manifold (Q, g) . Suppose that the geodesic flow of g is completely integrable. If $\text{reg } T^*Q$ intersects the space of horizontal vectors $\mathcal{H} \cong \Phi^{-1}(0)$ in a dense set then the geodesic flow on Q/G endowed with the submersion metric g_{sub} is completely integrable.*

Example 9.1 Eschenburg constructed bi-quotients $M_{k,l,p,q}^7 = SU(3)/T_{k,l,p,q}$ endowed with the submersion metrics $ds_{t,\text{sub}}^2$ with strictly positive sectional curvature. Here $T_{k,l,p,q} \cong T^1 \subset T^2 \times T^2$, where T^2 is a maximal torus and ds_t^2 is a one-parameter family of left-invariant metrics on $SU(3)$ (see [29]). One can prove that the geodesic flows of the metrics ds_t^2 are completely integrable and that we can apply Theorem 9.2 to get the integrability of the geodesic flows of the submersion metrics on $M_{k,l,p,q}^7$.

Remark 9.2 It is very interesting that on all known manifolds which admit metrics with strictly positive sectional curvatures (see Wilking [82]) one can find (positive sectional curvature) metrics with completely integrable geodesic flows.

Here is a simple construction that gives examples satisfying the hypotheses of Theorem 9.2. Suppose we are given Hamiltonian G -actions on two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) with moment maps Φ_{M_1} and Φ_{M_2} . Then we have the natural diagonal action of G on the product $(M_1 \times M_2, \omega_1 \oplus \omega_2)$, with moment map

$$\Phi_{M_1 \times M_2} = \Phi_{M_1} + \Phi_{M_2}. \quad (9.3)$$

If (Q_1, g_1) and (Q_2, g_2) have integrable geodesic flows, then $(Q_1 \times Q_2, g_1 \oplus g_2)$ also has integrable geodesic flow. Using (9.3), one can easily see that if the G -actions on Q_1 and Q_2 are almost everywhere locally free, and free on the product $Q_1 \times Q_2$ then a generic horizontal vector of the submersion $Q_1 \times Q_2 \rightarrow Q_1 \times_G Q_2 = (Q_1 \times Q_2)/G$ belongs to $\text{reg } T^*(Q_1 \times Q_2)$. Thus the geodesic flow on $Q_1 \times_G Q_2$, endowed with the submersion metric, is completely integrable.

Example 9.2 Suppose a compact Lie group G acts freely by isometries on (Q, g) . Let G_1 be an arbitrary compact Lie group, which contains G as a subgroup. Let ds_1^2 be some left-invariant Riemannian metric on G_1 with integrable geodesic flow. Then G acts in the natural way by isometries on (G_1, ds_1^2) . Therefore, if the geodesic flow on Q is completely integrable, then the geodesic flows on $Q \times_G Q$ and $Q \times_G G_1$ endowed with the submersion metrics will be also completely integrable.

10 Geodesic flows on the spheres

In this section we shall list some of the known interesting integrable geodesic flows on the spheres.

Submersion metrics. The sphere S^{n-1} and the cotangent bundle T^*S^{n-1} are given by the equations

$$\begin{aligned} S^{n-1} &= \{x \in \mathbb{R}^n, \langle q, q \rangle = 1\} \\ T^*S^{n-1} &= \{(x, y) \in \mathbb{R}^{2n} \mid \langle q, p \rangle = 1, \langle q, q \rangle = 1, \langle q, p \rangle = 0\}. \end{aligned}$$

Following section 7 to construct integrable geodesic flows on the sphere we can use the structure of a homogeneous space. This structure is not unique but one can start from the simplest one $S^{n-1} = SO(n)/SO(n-1)$. The moment map $\Phi : T^*S^{n-1} \rightarrow so(n) \cong so(n)^*$ of the natural $SO(n)$ action is then given by

$$\Phi(q, p) = q \wedge p = qp^t - pq^t.$$

To construct an integrable geodesic flow on S^{n-1} we can use an arbitrary integrable system on the Lie algebra $so(n)$ with quadratic positive definite hamiltonian (speaking more precisely such a system must be integrable on singular orbits lying in the image of the momentum mapping). There are several series and some exceptional examples of such systems (see, for instance, [51, 1, 2, 54]). The most famous among them is the Manakov integrable case [51]. Using it, Brailov obtained the following integrable geodesic flow [21, 22].

Let $a_1 > a_2 > \dots > a_n > a_{n+1}$, $b_{n+1} > b_1 > b_2 > \dots > b_n$.

Theorem 10.1 (Brailov, 1983) *The geodesic flow of the metric on S^{n-1} obtained by submersion from the right-invariant Manakov metric on $SO(n)$ with*

Hamiltonian function

$$H_{a,b}(q,p) = \frac{1}{2} \sum_{1 \leq i < j \leq n} \frac{b_i - b_j}{a_i - a_j} (q_i p_j - q_j p_i)^2$$

is completely integrable. Moreover, the deformation of the metric given by the Hamiltonian function $F(q,p) = H_{a,b} + \frac{1}{2} \sum_{1 \leq i \leq n} \frac{b_{n+1} - b_i}{a_i - a_{n+1}} p_i^2$ is also completely integrable. The latter has integrals

$$F_k = \sum_{1 \leq i < j \leq n} \frac{a_i^k - a_j^k}{a_i - a_j} (q_i p_j - q_j p_i)^2 - \sum_{1 \leq i \leq n} \frac{a_{n+1}^k - a_i^k}{a_{n+1} - a_i} p_i^2.$$

It can be checked that the metric which corresponds to $H_{a,b}$ with $b_i = -a_i^{-1}$ is

$$ds_a^2 = \frac{1}{\langle A^{-1}q, q \rangle} \langle Adx, dx \rangle, \quad A = \text{diag}(a_1, \dots, a_n). \quad (10.1)$$

For $n = 3$, the metric (10.1) is proportional to the metric on the Poisson sphere S^2 , i.e., to the metric obtained after $SO(2)$ reduction of the free rigid body motion around a fixed point with inertia tensor $I = A^{-1}$ (e.g., see [8])

Let us note that in the above construction one can use the other representations of S^{n-1} as a homogeneous space, for example, $S^{4n-1} = Sp(2n)/Sp(2n-1)$ and $S^6 = G_2/SU(3)$. It is an interesting question whether one really can construct in such a way new examples of integrable geodesic flows or the flows so obtained will be reduced in some sense to the above case $SO(n)/SO(n-1)$?

The Maupertuis principle and Neumann system. Consider the natural mechanical system with Hamiltonian $H(p,x) = \frac{1}{2} \sum g^{ij}(x) p_i p_j + V(x)$ on a compact Riemannian manifold (M, ds^2) . Here g^{ij} is the inverse of the metric tensor and $V(x)$ is a smooth potential on the configuration space M . Let h be greater than $\max V(x)$. By the classical *Maupertuis principle* the integral trajectories of the vector field X_H coincide (up to reparametrization) with the trajectories of another vector field X_{H_h} with Hamiltonian $H_h(x,p) = \frac{1}{2} \sum \frac{g^{ij}(x)}{h - V(x)} p_i p_j$ on the fixed iso-energy level $\mathcal{E}_h = \{H(x,p) = h\} = \{H_h(x,p) = 1\}$ (see [4, 11]). The Hamiltonian flow of H_h is the geodesic flow of the Riemannian metric

$$ds_h^2 = (h - V(x)) ds^2,$$

conformally equivalent to the original one ds^2 . Now, it is clear that if we start with an integrable system such that \mathcal{E}_h is almost everywhere foliated by invariant tori, the geodesic flow of the metric ds_h^2 will be completely integrable.

This idea can be used to construct non-trivial integrable geodesic flows on S^n starting from integrable potential systems on the standard sphere, ellipsoid or Poisson sphere. Such a potential systems have been studied, in particular, in [8, 83, 27, 39, 28].

For example, consider the Neumann system on the sphere [65], i.e., the motion of a mass point on the sphere S^{n-1} under the influence of the force with potential $V(q) = \frac{1}{2} \langle Aq, q \rangle$ (we take A as above):

$$H(q,p) = \frac{1}{2} \langle p, p \rangle + \frac{1}{2} \langle Aq, q \rangle.$$

The algebraic form of the integrals is found by K. Uhlenbeck [62, 63]:

$$F_k(q, p) = q_k^2 + \sum_{i \neq k} \frac{(q_k p_i - q_i p_k)^2}{a_k - a_i}.$$

Therefore, the geodesic flow of the metric $ds_h^2 = (h - \langle Aq, q \rangle) \langle dq, dq \rangle|_{S^{n-1}}$ is completely integrable. The integrals are given by (see [11])

$$\bar{F}_k(q, p) = \frac{\langle p, p \rangle}{2h - \langle Aq, q \rangle} q_k^2 + \sum_{i \neq k} \frac{(q_k p_i - q_i p_k)^2}{a_k - a_i}.$$

Note that there is a remarkable correspondence between the Neumann system and the geodesic flow on the ellipsoid via the Gauss mapping (Knörrer [44]).

Geodesical equivalence. After changing the coordinates $x_i = \sqrt{a_i} q_i$, the metric (10.1) take the form

$$ds_a^2 = \frac{1}{\langle A^{-1}x, A^{-1}x \rangle} \langle dx, dx \rangle|_Q, \quad (10.2)$$

conformally equivalent to the standard metric $ds^2 = \langle dx, dx \rangle$ of the ellipsoid $Q = \{\langle A^{-1}x, x \rangle = 1\}$. There is an interesting relation between these metrics from the point of view of geodesic equivalence. Namely, the standard metric is geodesically equivalent to the metric (see [52, 79]):

$$d\bar{s}^2 = \frac{1}{\langle A^{-1}x, A^{-1}x \rangle} \langle A^{-1}dx, dx \rangle|_Q.$$

Let g and \bar{g} be the corresponding metric tensors. Then one can define the operator $S = \left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{n}} \bar{g}^{-1}g$ and metrics $g_k = gS^k$, $\bar{g}_k = \bar{g}S^k$, $k \in \mathbb{Z}$. It appears that metrics g_k and \bar{g}_k are geodesically equivalent [79]. They are all separable in elliptic coordinates and have integrable geodesic flows. Explicit calculations shows [79]

$$S = (A - x \otimes x)|_Q$$

and, therefore, the metric \bar{g}_1 is given by (10.2). It is interesting that the Beltrami-Klein metric of the Lobachevsky space $\mathbb{H}^{n-1} = \{x \in \mathbb{R}^{n-1}, \frac{x_1^2}{a_1} + \dots + \frac{x_{n-1}^2}{a_{n-1}} \leq 1\}$ can be seen as a limit of the metric $d\bar{s}^2$ as the smallest semiaxis a_n of the ellipsoid Q tends to zero [28].

Kovalevskaya and Goryachev–Chaplygin metrics on the sphere S^2 . We already know that on the sphere S^2 we can find metrics with integrable geodesic flows by means of an integral polynomial in momenta of the first or second degree. The natural question is the existence of metrics with polynomial integral which can not be reduced to linear and quadratic ones. The positive answer for additional integrals of 3-th and 4-th degrees is given by Bolsinov and Fomenko with two examples: the Kovalevskaya ds_K^2 and Goryachev–Chaplygin ds_{GC}^2 metrics (see [11]).

The motion of a rigid body about a fixed point in the presence of the gravitation field admits $SO(2)$ -reduction (rotations about the direction of gravitational field). Taking the integrable Kovalevskaya and Goryachev–Chaplygin

cases we get the integrable systems on T^*S^2 . The metrics ds_K^2 and ds_{GC}^2 then can easily be constructed by means of the Maupertuis principle. They are the restrictions of the metrics

$$ds_K^2 = \frac{h - q_1}{2} \frac{\langle Adx, dx \rangle}{\langle A^{-1}q, q \rangle}, \quad A = \text{diag}(1, 1, 2),$$

$$ds_{GC}^2 = \frac{h - q_1}{4} \frac{\langle Adx, dx \rangle}{\langle A^{-1}q, q \rangle} \quad A = \text{diag}(1, 1, 4)$$

to the unit sphere. The geodesic flow of ds_K^2 and ds_{GC}^2 admits the first integrals of degree four and three in velocities, which can not be reduced to lower degrees (for more details see [11]).

New families of metrics with cubic and fourth degrees integrals are given by Selivanova [73, 74]. Just recently K.Kiyohara has constructed integrable geodesic flows with polynomial integrable of arbitrary degree $k > 2$ [42]. The idea of his construction is the following. First take the constant curvature metric ds_0^2 . Its geodesic flow admit three independent linear integrals. Take two of them f_1 and f_2 and consider the polynomial integral $P = f_1^l f_2^m$ of degree $k = m + l$. It appears one can perturb the metric ds_0^2 in such a way that its geodesic flows remains integrable and, moreover, the first integral \tilde{P} preserve its form, that is, $\tilde{P} = \tilde{f}_1^l \tilde{f}_2^m$ where \tilde{f}_1 and \tilde{f}_2 are linear functions (but not integrals anymore). However, the geodesic flow still has the property that all geodesics are closed with the same period.

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